Advantage Testing Foundation

The First Annual Math Prize for Girls

Saturday, November 14, 2009

TEST BOOKLET

Test Version A

DIRECTIONS

1. Do not open this test until your proctor instructs you to.
2. Fill out the top of your answer sheet. Your student number is on your name tag. Your test version (A, B, C, or D) is above on this page.
3. You may use pencils, pens, and erasers. You may also use the scratch paper that we provide. You may NOT use calculators, rulers, protractors, compasses, graph paper, books, or slide rules.
4. Write your final answers on the answer sheet. When a problem specifies a particular form for the answer, write your answer in that form. Write your answers clearly. We will collect only the answer sheet from you.

NOTES

1. This test contains 20 problems. You will have 150 minutes (2.5 hours) to take the test. Your score will be the number of correct answers.
2. Figures are not necessarily drawn to scale.
3. Good luck!

Math Prize for Girls 2009

1. How many ordered pairs of integers (x, y) are there such that

$$0 < |xy| < 36?$$

2. If a, b, c, d, and e are constants such that every $x > 0$ satisfies

$$\frac{5x^4 - 8x^3 + 2x^2 + 4x + 7}{(x+2)^4} = a + \frac{b}{x+2} + \frac{c}{(x+2)^2} + \frac{d}{(x+2)^3} + \frac{e}{(x+2)^4}\,,$$

then what is the value of $a + b + c + d + e$?

3. The *Fibonacci numbers* are defined recursively by the equation

$$F_n = F_{n-1} + F_{n-2}$$

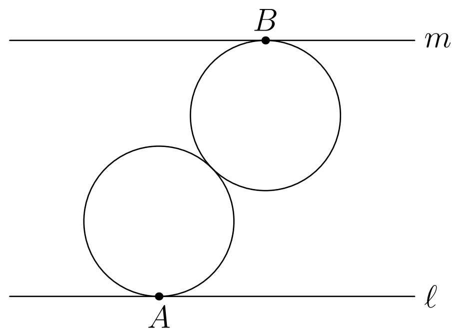
for every integer $n \geq 2$, with initial values $F_0 = 0$ and $F_1 = 1$. Let $G_n = F_{3n}$ be every third Fibonacci number. There are constants a and b such that every integer $n \geq 2$ satisfies

$$G_n = aG_{n-1} + bG_{n-2}.$$

Compute the ordered pair (a, b).

4. The admission fee for an exhibition is \$25 per adult and \$12 per child. Last Tuesday, the exhibition collected \$1950 in admission fees from at least one adult and at least one child. Of all the possible ratios of adults to children at the exhibition last Tuesday, which one is closest to 1? Express your answer as a fraction in reduced form.

5. The figure below shows two parallel lines, ℓ and m, that are distance 12 apart:



A circle is tangent to line ℓ at point A. Another circle is tangent to line m at point B. The two circles are congruent and tangent to each other as shown. The distance between A and B is 13. What is the radius of each circle? Express your answer as a fraction in reduced form.

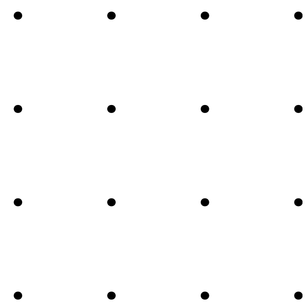
6. Consider a fair coin and a fair 6-sided die. The die begins with the number 1 face up. A *step* starts with a toss of the coin: if the coin comes out heads, we roll the die; otherwise (if the coin comes out tails), we do nothing else in this step. After 5 such steps, what is the probability that the number 1 is face up on the die? Express your answer as a fraction in reduced form.

7. Compute the value of the expression

$$2009^4 - 4 \times 2007^4 + 6 \times 2005^4 - 4 \times 2003^4 + 2001^4 \,.$$

8. Which point on the circle $(x - 11)^2 + (y - 13)^2 = 116$ is farthest from the point $(41, 25)$? Express your answer as an ordered pair.

9. The figure below is a 4×4 grid of points.



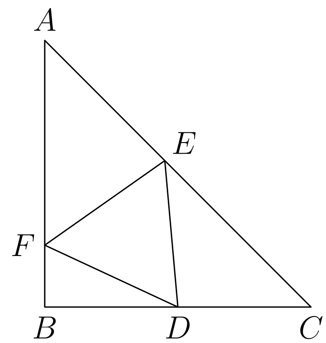
Each pair of horizontally adjacent or vertically adjacent points are distance 1 apart. In the plane of this grid, how many circles of radius 1 pass through exactly two of these grid points?

10. When the integer $\left(\sqrt{3}+5\right)^{103} - \left(\sqrt{3}-5\right)^{103}$ is divided by 9, what is the remainder?

11. An arithmetic sequence consists of 200 numbers that are each at least 10 and at most 100. The sum of the numbers is 10,000. Let L be the *least* possible value of the 50th term and let G be the *greatest* possible value of the 50th term. What is the value of $G - L$? Express your answer as a fraction in reduced form.

12. Jenny places 100 pennies on a table, 30 showing heads and 70 showing tails. She chooses 40 of the pennies at random (all different) and turns them over. That is, if a chosen penny was showing heads, she turns it to show tails; if a chosen penny was showing tails, she turns it to show heads. At the end, what is the expected number (average number) of pennies showing heads?

13. The figure below shows a right triangle $\triangle ABC$.



The legs $\overline{AB}$ and $\overline{BC}$ each have length 4. An equilateral triangle $\triangle DEF$ is inscribed in $\triangle ABC$ as shown. Point D is the midpoint of $\overline{BC}$. What is the area of $\triangle DEF$? Express your answer in the form $m\sqrt{3} - n$, where m and n are positive integers.

14. The three roots of the cubic $30x^3 - 50x^2 + 22x - 1$ are distinct real numbers between 0 and 1. For every nonnegative integer n, let s_n be the sum of the nth powers of these three roots. What is the value of the infinite series

$$s_0 + s_1 + s_2 + s_3 + \dots ?$$

15. Let $x = \sqrt[3]{\frac{4}{25}}$. There is a unique value of y such that $0 < y < x$ and $x^x = y^y$. What is the value of y? Express your answer in the form $\sqrt[c]{\frac{a}{b}}$, where a and b are relatively prime positive integers and c is a prime number.

16. Let x be a real number such that the five numbers $\cos(2\pi x)$, $\cos(4\pi x)$, $\cos(8\pi x)$, $\cos(16\pi x)$, and $\cos(32\pi x)$ are all nonpositive. What is the smallest possible positive value of x? Express your answer as a fraction in reduced form.

17. Let a, b, c, x, y, and z be real numbers that satisfy the three equations

$$\begin{aligned} 13x + by + cz &= 0 \\ ax + 23y + cz &= 0 \\ ax + by + 42z &= 0. \end{aligned}$$

Suppose that $a \neq 13$ and $x \neq 0$. What is the value of

$$\frac{13}{a-13} + \frac{23}{b-23} + \frac{42}{c-42}\,?$$

18. The value of 21! is 51,090,942,171,abc,440,000, where a, b, and c are digits. What is the value of $100a + 10b + c$?

19. Let S be a set of 100 points in the plane. The distance between every pair of points in S is different, with the largest distance being 30. Let A be one of the points in S, let B be the point in S farthest from A, and let C be the point in S farthest from B. Let d be the distance between B and C rounded to the nearest integer. What is the smallest possible value of d?

20. Let y_0 be chosen randomly from $\{0, 50\}$, let y_1 be chosen randomly from $\{40, 60, 80\}$, let y_2 be chosen randomly from $\{10, 40, 70, 80\}$, and let y_3 be chosen randomly from $\{10, 30, 40, 70, 90\}$. (In each choice, the possible outcomes are equally likely to occur.) Let P be the unique polynomial of degree less than or equal to 3 such that $P(0) = y_0$, $P(1) = y_1$, $P(2) = y_2$, and $P(3) = y_3$. What is the expected value of $P(4)$?