MATH PRIZE *for* GIRLS

2009 Solutions

Problem 1 How many ordered pairs of integers (x, y) are there such that

$$0 < |xy| < 36?$$

Answer: 524.

Solution: We will first assume that x and y are positive. Either $x < y$, $x = y$, or $x > y$.

If $x = y$, we get the inequality $x^2 < 36$. So x is 1, 2, 3, 4, or 5. We have 5 solutions: $(1, 1)$, $(2, 2)$, $(3, 3)$, $(4, 4)$, and $(5, 5)$.

Next suppose that $x < y$. Then again we get $x^2 < xy < 36$. So x is 1, 2, 3, 4, or 5. If $x = 1$, then y can go from 2 through 35: 34 solutions. If $x = 2$, then y can go from 3 through 17: 15 solutions. If $x = 3$, then y can go from 4 through 11: 8 solutions. If $x = 4$, then y can go from 5 through 8: 4 solutions. If $x = 5$, then y can go from 6 through 7: 2 solutions. So the total number of solutions in this case is 63.

If $x > y$, then by symmetry there are also 63 solutions.

So adding all three cases gives $5 + 63 + 63 = 131$.

The analysis so far was under the assumption that x and y are positive. By symmetry, there are the same number of solutions for all four choices of signs (positive-positive, positive-negative, negative-positive, and negative-negative). So the total number of solutions is $4 \cdot 131 = \boxed{524}$.

Problem 2 If a, b, c, d, and e are constants such that every $x > 0$ satisfies

$$\frac{5x^4 - 8x^3 + 2x^2 + 4x + 7}{(x+2)^4} = a + \frac{b}{x+2} + \frac{c}{(x+2)^2} + \frac{d}{(x+2)^3} + \frac{e}{(x+2)^4},$$

then what is the value of $a + b + c + d + e$?

Answer: 18.

Solution: If we multiply the equation by $(x+2)^4$, we will get polynomials on both sides. Because the two polynomials are equal for all $x > 0$ (an infinite number of values), they must be equal for *all* x. Dividing by $(x+2)^4$, we see that our original equation must be true for all $x \neq -2$. Note that if we plug in $x = -1$, the right side becomes $a+b+c+d+e$, which is what we want. If we plug in $x = -1$ on the left side, we get

$$\begin{aligned} a+b+c+d+e &= 5(-1)^4 - 8(-1)^3 + 2(-1)^2 + 4(-1) + 7 \\ &= 5 + 8 + 2 - 4 + 7 \\ &= \boxed{18}. \end{aligned}$$

Problem 3 The *Fibonacci numbers* are defined recursively by the equation

$$F_n = F_{n-1} + F_{n-2}$$

for every integer $n \geq 2$, with initial values $F_0 = 0$ and $F_1 = 1$. Let $G_n = F_{3n}$ be every third Fibonacci number. There are constants a and b such that every integer $n \geq 2$ satisfies

$$G_n = aG_{n-1} + bG_{n-2}.$$

Compute the ordered pair (a, b).

Answer: $(4, 1)$.

Solution: Besides $F_0 = 0$ and $F_1 = 1$, we know that $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, and $F_9 = 34$. Thus we have $G_0 = 0$, $G_1 = 2$, $G_2 = 8$, and $G_3 = 34$. Plugging $n = 2$ into the equation $G_n = aG_{n-1} + bG_{n-2}$, we get $8 = 2a$, so $a = 4$. Plugging in $n = 3$, we get $34 = 8a + 2b$, so $b = 1$. Hence the ordered pair is $\boxed{(4, 1)}$.

Alternative Solution: (due to Kouichi Nakagawa) Applying the Fibonacci recurrence many times, we have

$$\begin{aligned} G_n &= F_{3n} \\ &= F_{3n-1} + F_{3n-2} \\ &= 2F_{3n-2} + F_{3n-3} \\ &= 3F_{3n-3} + 2F_{3n-4} \\ &= 3F_{3n-3} + F_{3n-4} + F_{3n-5} + F_{3n-6} \\ &= 4F_{3n-3} + F_{3n-6} \\ &= 4G_{n-1} + G_{n-2}. \end{aligned}$$

So $(a, b) = \boxed{(4, 1)}$.

Problem 4 The admission fee for an exhibition is \$25 per adult and \$12 per child. Last Tuesday, the exhibition collected \$1950 in admission fees from at least one adult and at least one child. Of all the possible ratios of adults to children at the exhibition last Tuesday, which one is closest to 1? Express your answer as a fraction in reduced form.

Answer: $\dfrac{27}{25}$.

Solution: Let a be the number of adults and c be the number of children. Then we get the condition

$$25a + 12c = 1950 = 25 \cdot 78.$$

Rearranging gives

$$12c = 25(78 - a).$$

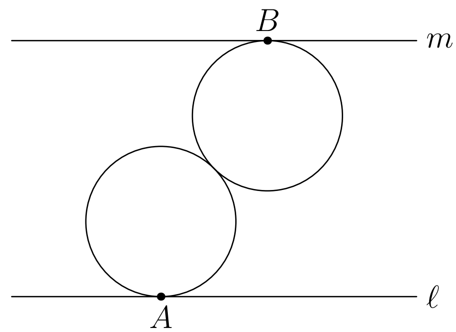
So c must be a multiple of 25.

We will make a chart of the multiples of 25. Note that as c goes up by 25, a goes down by 12.

a	c
66	25
54	50
42	75
30	100
18	125
6	150

By inspection, the ratio of adults to children closest to 1 is $\frac{54}{50}$, or $\boxed{\dfrac{27}{25}}$.

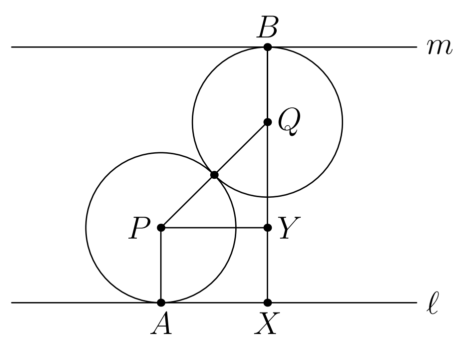
Problem 5 The figure below shows two parallel lines, ℓ and m, that are distance 12 apart:



A circle is tangent to line ℓ at point A. Another circle is tangent to line m at point B. The two circles are congruent and tangent to each other as shown. The distance between A and B is 13. What is the radius of each circle? Express your answer as a fraction in reduced form.

Answer: $\dfrac{169}{48}$.

Solution:



As in the figure above, draw the line perpendicular to line ℓ passing through point B. Let X be the intersection point of this line and ℓ. By the Pythagorean Theorem applied to $\triangle AXB$, we have $AX = \sqrt{13^2 - 12^2} = 5$. Let P be the center of the circle passing through A and let Q be the center of the circle passing through B. Let r be the radius of each circle. Draw the line parallel to ℓ passing through P. Let Y be the intersection point of this line and $\overline{XB}$.

Due to the right angles, $AXYP$ is a rectangle and PYQ is a right triangle. So $PY = 5$ and $XY = r$. Hence $QY = 12 - 2r$. Also $PQ = 2r$. By the

Pythagorean Theorem, we get the equation

$$5^2 + (12 - 2r)^2 = (2r)^2.$$

Expanding both sides, we get

$$25 + 144 - 48r + 4r^2 = 4r^2.$$

The quadratic terms cancel:

$$25 + 144 - 48r = 0.$$

Solving for r gives $r = \boxed{\dfrac{169}{48}}$.

Problem 6 Consider a fair coin and a fair 6-sided die. The die begins with the number 1 face up. A *step* starts with a toss of the coin: if the coin comes out heads, we roll the die; otherwise (if the coin comes out tails), we do nothing else in this step. After 5 such steps, what is the probability that the number 1 is face up on the die? Express your answer as a fraction in reduced form.

Answer: $\dfrac{37}{192}$.

Solution: If all the coin tosses come out tails, then the die will remain at 1. The probability that all 5 tosses will be tails is $\frac{1}{2^5} = \frac{1}{32}$.

Otherwise, if at least one of the tosses comes out heads, then at the end the die will be in a random position, because each roll has an equal probability of showing any particular number. The probability that at least one toss is heads is $1 - \frac{1}{32}$, or $\frac{31}{32}$. Given that condition, the probability that the die will be 1 is $\frac{1}{6}$. So the probability that at least one toss is heads and the die ends at 1 is $\frac{31}{32} \cdot \frac{1}{6} = \frac{31}{192}$.

Hence, adding the two cases together, we get that the total probability is $\frac{1}{32} + \frac{31}{192}$, which evaluates to $\boxed{\dfrac{37}{192}}$.

Problem 7 Compute the value of the expression

$$2009^4 - 4 \times 2007^4 + 6 \times 2005^4 - 4 \times 2003^4 + 2001^4\,.$$

Answer: 384.

Solution: Let $f(x) = (2x+1)^4$. Then our problem is to evaluate

$$f(1004) - 4f(1003) + 6f(1002) - 4f(1001) + f(1000).$$

That expression is the fourth finite difference of f evaluated at 1000.

Note that f itself is a quartic polynomial with leading coefficient 16. So the first difference of f is a cubic with leading coefficient 64. The second difference is a quadratic with leading coefficient 192. The third difference is a linear function with leading coefficient 384. The fourth difference is the constant $\boxed{384}$.

Alternative Solution: Let $x = 2005$. Then our expression becomes

$$(x+4)^4 - 4(x+2)^4 + 6x^4 - 4(x-2)^4 + (x-4)^4.$$

We can simplify by first applying the Binomial Theorem, and then adding the first and fifth terms together and adding the second and fourth terms together. We get that the expression equals

$$2\left(x^4 + 6 \cdot 4^2 x^2 + 4^4\right) - 4 \cdot 2\left(x^4 + 6 \cdot 2^2 x^2 + 2^4\right) + 6 \cdot x^4.$$

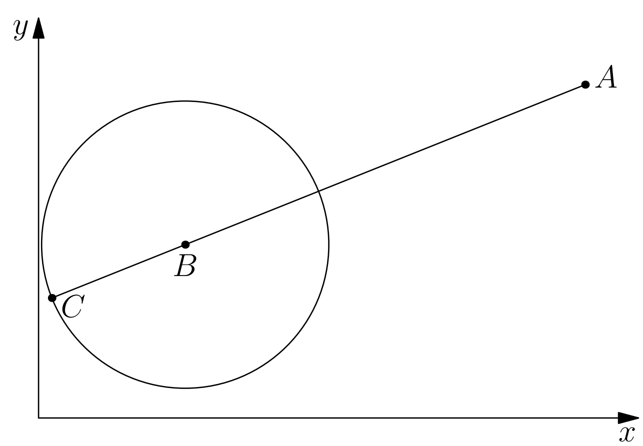
The x^4 and x^2 terms conveniently cancel out. So we are left with

$$2 \cdot 4^4 - 8 \cdot 2^4 = 512 - 128 = \boxed{384}.$$

Problem 8 Which point on the circle $(x-11)^2 + (y-13)^2 = 116$ is farthest from the point $(41, 25)$? Express your answer as an ordered pair.

Answer: $(1, 9)$.

Solution:

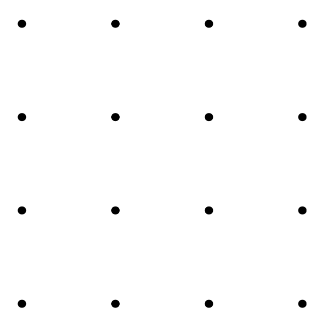


In the figure above, let A be the point $(41, 25)$. Let $B = (11, 13)$ be the center of the circle. The line $\overleftrightarrow{AB}$ intersects the circle in two points. Let C be the intersection point that is farther from A. The radius of the circle is $2\sqrt{29}$. The distance between A and B is $6\sqrt{29}$.

By the triangle inequality, every point on the circle is at most distance $8\sqrt{29}$ from A. The point C achieves that distance, so it is the point we are looking for.

Because $BC = 2\sqrt{29}$ and $BA = 6\sqrt{29}$, the vector $\overrightarrow{BC}$ is $\frac{1}{3}$ of vector $\overrightarrow{BA}$. Because the vector $\overrightarrow{BA}$ is $(30, 12)$, the vector $\overrightarrow{CB}$ is $(10, 4)$. Hence the coordinates of C are $\boxed{(1, 9)}$.

Problem 9 The figure below is a 4×4 grid of points.



Each pair of horizontally adjacent or vertically adjacent points are distance 1 apart. In the plane of this grid, how many circles of radius 1 pass through exactly two of these grid points?

Answer: 52.

Solution: For each pair of horizontally adjacent points, there are two circles that go through them and none of the other grid points. This claim can be verified by looking at the intersection of the two circles of radius 1 centered at those two points; the two intersection points are the centers of the two circles that go through the two points. The number of horizontally adjacent pairs is $4 \cdot 3 = 12$. So there are 24 circles in this case.

The same is true for vertically adjacent points. So we get another 24 circles.

Finally, we can draw a circle at each corner point. That's 4 more circles. The total is $24 + 24 + 4 = \boxed{52}$.

Problem 10 When the integer $\left(\sqrt{3}+5\right)^{103} - \left(\sqrt{3}-5\right)^{103}$ is divided by 9, what is the remainder?

Answer: 1.

Solution: By the Binomial Theorem, we have

$$\left(5+\sqrt{3}\right)^{103} = 5^{103} + 103\cdot 5^{102}\sqrt{3} + \binom{103}{2}5^{101}\cdot 3 + \binom{103}{3}5^{100}\cdot 3\sqrt{3} + \cdots + 103\cdot 5\cdot 3^{51} + \sqrt{3}^{103}.$$

Similarly, we have

$$\left(5-\sqrt{3}\right)^{103} = 5^{103} - 103\cdot 5^{102}\sqrt{3} + \binom{103}{2}5^{101}\cdot 3 - \binom{103}{3}5^{100}\cdot 3\sqrt{3} + \cdots + 103\cdot 5\cdot 3^{51} - \sqrt{3}^{103}.$$

Adding, we get

$$\left(5+\sqrt{3}\right)^{103} + \left(5-\sqrt{3}\right)^{103} = 2\cdot 5^{103} + 2\binom{103}{2}5^{101}\cdot 3 + \cdots + 2\cdot 103\cdot 5\cdot 3^{51}.$$

On the right side, every term besides the first two are multiples of 9. So modulo 9, we have

$$\left(5+\sqrt{3}\right)^{103} + \left(5-\sqrt{3}\right)^{103} \equiv 2\cdot 5^{103} + 2\binom{103}{2}5^{101}\cdot 3 \pmod{9}.$$

Because $5^3 = 125 \equiv -1 \pmod{9}$, we have $5^6 \equiv 1 \pmod{9}$. (We could also see that by Euler's totient theorem.) Thus $5^{103} = 5\cdot 5^{102} \equiv 5 \pmod{9}$. Also $\binom{103}{2}$ is divisible by 3, so $2\binom{103}{2}5^{101}\cdot 3$ is divisible by 9. Thus our total mod 9 is

$$2\cdot 5^{103} + 2\binom{103}{2}5^{101}\cdot 3 \equiv 2\cdot 5 + 0 = 10 \equiv 1 \pmod{9}.$$

So the remainder is $\boxed{1}$.

Problem 11 An arithmetic sequence consists of 200 numbers that are each at least 10 and at most 100. The sum of the numbers is 10,000. Let L be the *least* possible value of the 50th term and let G be the *greatest* possible value of the 50th term. What is the value of $G - L$? Express your answer as a fraction in reduced form.

Answer: $\dfrac{8080}{199}$.

Solution: The 200 numbers have sum 10,000, so their average is $\frac{10{,}000}{200} = 50$. Hence we can represent the sequence in the form

$$50 - 199d, 50 - 197d, \ldots, 50 - d, 50 + d, \ldots, 50 + 197d, 50 + 199d.$$

Because all the terms are at least 10 (in particular, the first and last terms), we have

$$199\,|d| \le 50 - 10 = 40.$$

Solving for d gives

$$|d| \le \frac{40}{199}.$$

The 50th term is $50 - 101d$. From the bound above, we have

$$101\,|d| \le 101 \cdot \frac{40}{199} = \frac{4040}{199}.$$

Hence the 50th term is between $50 - \frac{4040}{199}$ and $50 + \frac{4040}{199}$. By setting $d = \frac{40}{199}$ we get a sequence that achieves the lower bound and meets all the conditions of the problem; so $L = 50 - \frac{4040}{199}$. By setting $d = -\frac{40}{199}$ we get a sequence that achieves the upper bound and meets all the conditions of the problem; so $G = 50 + \frac{4040}{199}$. Hence $G - L = \boxed{\dfrac{8080}{199}}$.

Note: Because each term is at least 10 and their average is 50, each term is at most 90. So the condition that each term is at most 100 was redundant.

Problem 12 Jenny places 100 pennies on a table, 30 showing heads and 70 showing tails. She chooses 40 of the pennies at random (all different) and turns them over. That is, if a chosen penny was showing heads, she turns it to show tails; if a chosen penny was showing tails, she turns it to show heads. At the end, what is the expected number (average number) of pennies showing heads?

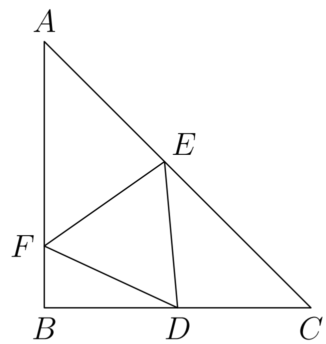
Answer: 46.

Solution: Let's first focus on a penny that started by showing tails. The probability that it turned to heads is $\frac{40}{100}$, or $\frac{2}{5}$. Since there are 70 such pennies, the average number of pennies that switched from tails to heads is $70 \cdot \frac{2}{5} = 28$.

Now let's focus on a penny that started off as heads. The probability that it stayed heads is $1 - \frac{2}{5}$, or $\frac{3}{5}$. There are 30 such pennies, so the average number of pennies that started and ended with heads is $30 \cdot \frac{3}{5} = 18$.

The average number of pennies that ended with heads is the sum of the two averages above: $28 + 18 = \boxed{46}$.

Problem 13 The figure below shows a right triangle $\triangle ABC$.



The legs $\overline{AB}$ and $\overline{BC}$ each have length 4. An equilateral triangle $\triangle DEF$ is inscribed in $\triangle ABC$ as shown. Point D is the midpoint of $\overline{BC}$. What is the area of $\triangle DEF$? Express your answer in the form $m\sqrt{3} - n$, where m and n are positive integers.

Answer: $22\sqrt{3} - 36$.

Solution: We will solve the problem with complex numbers. Let's impose a coordinate system where D is the origin, C is the real number 2, B is the real number -2, and A is $-2 + 4i$. Then F is of the form $-2 + bi$ for some real number b. Because E is a $60°$ *clockwise* rotation from F, we have

$$\begin{aligned} E &= F\left[\cos(-60°) + i\sin(-60°)\right] \\ &= (-2 + bi)\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ &= -1 + b\frac{\sqrt{3}}{2} + \left(\frac{1}{2}b + \sqrt{3}\right)i. \end{aligned}$$

Because E lies on the line $\overleftrightarrow{AC}$, its real and imaginary parts must add up to 2. So we get

$$-1 + b\frac{\sqrt{3}}{2} + \frac{1}{2}b + \sqrt{3} = 2.$$

Solving for b, we get

$$b = \frac{6 - 2\sqrt{3}}{\sqrt{3}+1} = \left(3 - \sqrt{3}\right)\left(\sqrt{3} - 1\right) = 4\sqrt{3} - 6.$$

The side length squared of the equilateral triangle is thus

$$|F|^2 = 2^2 + b^2 = 4 + \left(4\sqrt{3} - 6\right)^2 = 88 - 48\sqrt{3}.$$

Hence the area of the equilateral triangle is

$$\frac{\left(88 - 48\sqrt{3}\right)\sqrt{3}}{4} = \left(22 - 12\sqrt{3}\right)\sqrt{3} = \boxed{22\sqrt{3} - 36}.$$

Problem 14 The three roots of the cubic $30x^3 - 50x^2 + 22x - 1$ are distinct real numbers between 0 and 1. For every nonnegative integer n, let s_n be the sum of the nth powers of these three roots. What is the value of the infinite series

$$s_0 + s_1 + s_2 + s_3 + \dots?$$

Answer: 12.

Solution: Let p, q, and r be the roots. Then $s_n = p^n + q^n + r^n$. When we add over all n, we need to find the sum of p^n (and q^n and r^n). By the sum of a geometric series, that is $\frac{1}{1-p}$. So we need to evaluate

$$\frac{1}{1-p} + \frac{1}{1-q} + \frac{1}{1-r}.$$

The numbers $1-p$, $1-q$, and $1-r$ are the roots of the cubic $30(1-x)^3 - 50(1-x)^2 + 22(1-x) - 1$. The constant coefficient of that cubic is $30 - 50 + 22 - 1 = 1$. The linear coefficient of that cubic is $30(-3) + 50(2) - 22 = -12$. Thus $\frac{1}{1-p}$, $\frac{1}{1-q}$, and $\frac{1}{1-r}$ are the roots of a cubic in the reversed form $1x^3 - 12x^2 + \cdots$. So the sum of $\frac{1}{1-p}$, $\frac{1}{1-q}$, and $\frac{1}{1-r}$ is $-\frac{-12}{1} = \boxed{12}$.

Problem 15 Let $x = \sqrt[3]{\frac{4}{25}}$. There is a unique value of y such that $0 < y < x$ and $x^x = y^y$. What is the value of y? Express your answer in the form $\sqrt[c]{\frac{a}{b}}$, where a and b are relatively prime positive integers and c is a prime number.

Answer: $\sqrt[3]{\dfrac{32}{3125}}$.

Solution: We can rewrite x as

$$x = \sqrt[3]{\frac{4}{25}} = \left(\frac{4}{25}\right)^{1/3} = \left(\frac{2}{5}\right)^{2/3}.$$

The equation $x^x = y^y$ is equivalent to $x \log x = y \log y$. We can rewrite $x \log x$ as

$$\begin{aligned} x \log x &= \left(\frac{2}{5}\right)^{2/3} \cdot \frac{2}{3} \log \frac{2}{5} \\ &= \left(\frac{2}{5}\right)^{2/3} \cdot \frac{2}{5} \cdot \frac{5}{3} \log \frac{2}{5} \\ &= \left(\frac{2}{5}\right)^{5/3} \cdot \frac{5}{3} \log \frac{2}{5} \\ &= \left(\frac{2}{5}\right)^{5/3} \log \left(\frac{2}{5}\right)^{5/3}. \end{aligned}$$

By inspecting the right side, we see that $y = \left(\frac{2}{5}\right)^{5/3}$ satisfies the equation $x \log x = y \log y$. Furthermore $0 < y < x$. So we have found the unique value we were looking for. To express it in the form we were asked for, we have

$$y = \left(\frac{2}{5}\right)^{5/3} = \left(\frac{2^5}{5^5}\right)^{1/3} = \left(\frac{32}{3125}\right)^{1/3} = \boxed{\sqrt[3]{\frac{32}{3125}}}.$$

Problem 16 Let x be a real number such that the five numbers $\cos(2\pi x)$, $\cos(4\pi x)$, $\cos(8\pi x)$, $\cos(16\pi x)$, and $\cos(32\pi x)$ are all nonpositive. What is the smallest possible positive value of x? Express your answer as a fraction in reduced form.

Answer: $\dfrac{21}{64}$.

Solution: If $0 < x < \frac{1}{4}$, then $0 < 2\pi x < \frac{\pi}{2}$, which implies $\cos(2\pi x) > 0$, which is false. So $x \geq \frac{1}{4}$.

If $\frac{1}{4} \leq x < \frac{5}{16}$, then $2\pi \leq 8\pi x < 2\pi + \frac{\pi}{2}$, which implies $\cos(8\pi x) > 0$, which is false. So $x \geq \frac{5}{16}$.

If $\frac{5}{16} \leq x < \frac{21}{64}$, then $10\pi \leq 32\pi x < 10\pi + \frac{\pi}{2}$, which implies $\cos(32\pi x) > 0$, which is false. So $x \geq \frac{21}{64}$.

If $x = \frac{21}{64}$, then $2\pi x = \frac{21}{32}\pi$, $4\pi x = \frac{21}{16}\pi$, $8\pi x = 2\pi + \frac{5}{8}\pi$, $16\pi x = 4\pi + \frac{5}{4}\pi$, and $32\pi x = 10\pi + \frac{1}{2}\pi$. All have nonpositive cosines. So the smallest possible value of x is $\boxed{\dfrac{21}{64}}$.

Alternative Solution: Consider the binary (base 2) representation of the positive number x. Look at the bits (digits) after the radix point (the dot or the "decimal" point). Let's number them Bits 1, 2, 3, and so on. Then $\cos(2\pi x)$ is nonpositive if and only if Bits 1 and 2 are different. (That's because cosine is nonpositive in the 2nd and 3rd quadrants.) Similarly, the five given cosines are nonpositive if and only if Bits 1, 2, 3, 4, 5, and 6 alternate in value. So, to make x as small as possible, we should choose $x = (.010101)_2$. As a fraction, that number is $\frac{1}{4} + \frac{1}{16} + \frac{1}{64}$, or $\boxed{\dfrac{21}{64}}$.

Problem 17 Let a, b, c, x, y, and z be real numbers that satisfy the three equations

$$\begin{aligned} 13x + by + cz &= 0 \\ ax + 23y + cz &= 0 \\ ax + by + 42z &= 0. \end{aligned}$$

Suppose that $a \neq 13$ and $x \neq 0$. What is the value of

$$\frac{13}{a-13} + \frac{23}{b-23} + \frac{42}{c-42}\,?$$

Answer: -2.

Solution: We can rewrite the first equation as

$$(a-13)x = ax + by + cz.$$

Because $a \neq 13$ and $x \neq 0$, both sides of the equation are nonzero. Similarly we can rewrite the second and third equations as

$$\begin{aligned}(b-23)y &= ax+by+cz\\(c-42)z &= ax+by+cz.\end{aligned}$$

Again, both sides of both equations are nonzero. Solving for x, y, and z, we get

$$\begin{aligned}x &= \frac{ax+by+cz}{a-13}\\y &= \frac{ax+by+cz}{b-23}\\z &= \frac{ax+by+cz}{c-42}.\end{aligned}$$

Plugging those values back into the expression $ax+by+cz$, we have

$$\begin{aligned}ax+by+cz &= a\cdot\frac{ax+by+cz}{a-13}+b\cdot\frac{ax+by+cz}{b-23}+c\cdot\frac{ax+by+cz}{c-42}\\&=(ax+by+cz)\left[\frac{a}{a-13}+\frac{b}{b-23}+\frac{c}{c-42}\right].\end{aligned}$$

Cancelling the nonzero $ax+by+cz$ on both sides gives

$$\frac{a}{a-13}+\frac{b}{b-23}+\frac{c}{c-42}=1.$$

Hence we have

$$\begin{aligned}\frac{13}{a-13}+\frac{23}{b-23}+\frac{42}{c-42} &= \left(\frac{a}{a-13}-1\right)+\left(\frac{b}{b-23}-1\right)+\left(\frac{c}{c-42}-1\right)\\&=\frac{a}{a-13}+\frac{b}{b-23}+\frac{c}{c-42}-3\\&=1-3\\&=\boxed{-2}.\end{aligned}$$

Problem 18 The value of 21! is 51,090,942,171,abc,440,000, where a, b, and c are digits. What is the value of $100a+10b+c$?

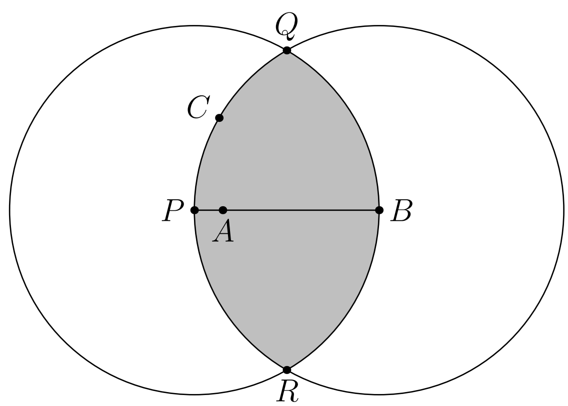
Answer: 709.

Solution: The number 21! is divisible by the primes 7, 11, and 13, so it is divisible by their product 1001. The divisibility test for 1001 is similar to that for 11, except that we group digits into blocks of 3. (That works because 1001 is 1 more than 1000.) So we know that $000 - 440 + abc - 171 + 942 - 090 + 51$ is divisible by 1001. In other words, $abc + 292$ is divisible by 1001. The only way that can be is if abc is $1001 - 292 = 709$. So a is 7, b is 0, and c is 9. The answer is $\boxed{709}$.

Problem 19 Let S be a set of 100 points in the plane. The distance between every pair of points in S is different, with the largest distance being 30. Let A be one of the points in S, let B be the point in S farthest from A, and let C be the point in S farthest from B. Let d be the distance between B and C rounded to the nearest integer. What is the smallest possible value of d?

Answer: 17.

Solution:



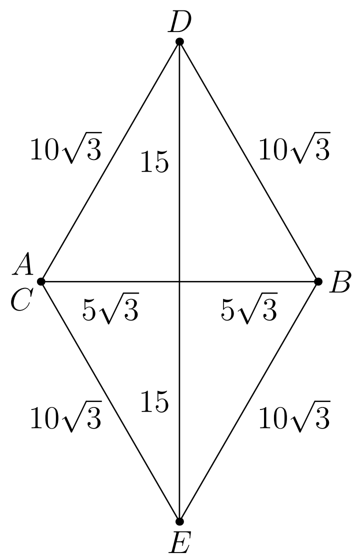
Let $r = BC$. Consider the circle above on the right with center B and radius r. Because C is the point in S farthest from B, every point in S is in or on that circle.

Let P be the point on that circle closest to point A. Every point in S is at most distance AB from A and hence (by the triangle inequality) is at most distance $PA + AB$ from P. But $PA + AB = PB = r$. So every point in S is in or on the circle above on the left with center P and radius r.

Consider the lens above that is the intersection of the two disks of radius r centered at B and P. By our previous work, every point in S is in that lens. The two points in the lens that are farthest apart are the points Q

and R. Because PBQ and PBR are equilateral triangles, we can calculate that $QR = r\sqrt{3}$. So we get the inequality $r\sqrt{3} \geq 30$. Solving for r, we find $r \geq 10\sqrt{3} \geq 17$. So d, which is r rounded to the nearest integer, is at least 17.

To finish, we need a configuration in which $d = 17$. Let's temporarily ignore the condition that all the distances be different. Look at the following configuration.



In this configuration, the points A and C coincide. The set S will consist of the five points A, B, C, D, and E, plus 95 points near the center. The largest distance is $DE = 30$, as desired. The point B is a point in S farthest from A, and C is a point in S farthest from B. The distance BC is $10\sqrt{3}$.

The only thing wrong with that configuration is that some of the distances are equal. But we can perturb the points A, B, and C a little away from the rhombus so that all the distances are different and so that BC is less than $10\sqrt{3} + 0.1$. In that case, BC rounded to the nearest integer will still be 17.

Putting both parts together, we see that the smallest possible value of d is $\boxed{17}$.

Problem 20 Let y_0 be chosen randomly from $\{0, 50\}$, let y_1 be chosen randomly from $\{40, 60, 80\}$, let y_2 be chosen randomly from $\{10, 40, 70, 80\}$, and let y_3 be chosen randomly from $\{10, 30, 40, 70, 90\}$. (In each choice, the possible outcomes are equally likely to occur.) Let P be the unique polynomial of degree less than or equal to 3 such that $P(0) = y_0$, $P(1) = y_1$, $P(2) = y_2$, and $P(3) = y_3$. What is the expected value of $P(4)$?

Answer: 107.

Solution: Let Q be the expected value of the polynomial P. Then Q is also a polynomial of degree less than or equal to 3. Writing E for expected value, we have

$$Q(0) = E(P(0)) = E(y_0) = \frac{0+50}{2} = \frac{50}{2} = 25.$$

Similarly, we have

$$\begin{aligned} Q(1) = E(P(1)) = E(y_1) &= \frac{40+60+80}{3} = \frac{180}{3} = 60 \\ Q(2) = E(P(2)) = E(y_2) &= \frac{10+40+70+80}{4} = \frac{200}{4} = 50 \\ Q(3) = E(P(3)) = E(y_3) &= \frac{10+30+40+70+90}{5} = \frac{240}{5} = 48. \end{aligned}$$

We are trying to find $Q(4)$. Because Q has degree less than or equal to 3, its fourth finite difference is zero. In particular, we have

$$Q(4) - 4Q(3) + 6Q(2) - 4Q(1) + Q(0) = 0.$$

Solving for $Q(4)$, we get

$$\begin{aligned} Q(4) &= 4Q(3) - 6Q(2) + 4Q(1) - Q(0) \\ &= 4 \cdot 48 - 6 \cdot 50 + 4 \cdot 60 - 25 \\ &= 192 - 300 + 240 - 25 \\ &= 432 - 325 \\ &= \boxed{107}. \end{aligned}$$