MATH PRIZE *for* GIRLS

2010 Olympiad Solutions

Problem 1 Let S be a set of 100 integers. Suppose that for all positive integers x and y (possibly equal) such that $x + y$ is in S, either x or y (or both) is in S. Prove that the sum of the numbers in S is at most 10,000.

Solution: Let the 100 integers of S be called a_1, a_2, …, a_{100}, where

$$a_1 < a_2 < \cdots < a_{100}\,.$$

Consider a particular index k (where $1 \le k \le 100$). We claim that $a_k \le 2k - 1$. Assume not; that is, assume $a_k \ge 2k$. Consider the following k sums:

$$\begin{aligned} 1 + (a_k - 1) &= a_k \\ 2 + (a_k - 2) &= a_k \\ &\vdots \\ k + (a_k - k) &= a_k. \end{aligned}$$

Because $a_k \ge 2k$, each term on the left appears only once (except that the two terms k and $a_k - k$ on the bottom left side might be equal). From the given condition, either 1 or $a_k - 1$ is in S; either 2 or $a_k - 2$ is in S; and on until either k or $a_k - k$ is in S. Hence, at least k numbers in S are less than a_k. But only $k - 1$ numbers in S are less than a_k (namely, a_1, a_2, …, a_{k-1}). Thus we have reached a contradiction. Therefore, we have proved that $a_k \le 2k - 1$.

From the bound $a_k \le 2k - 1$, we get the inequality

$$\sum_{k=1}^{100} a_k \le \sum_{k=1}^{100} (2k - 1)\,.$$

The right side is the sum of the first 100 odd numbers. It is well-known that the sum of the first 100 odd numbers is 100^2, which is 10,000. Hence, we are done.

Problem 2 Prove that for every positive integer n, there exist integers a and b such that $4a^2 + 9b^2 - 1$ is divisible by n.

Solution: First, we handle the case when n is odd. In that case, let b be 0 and let a be any integer satisfying $2a \equiv 1 \bmod n$. Then we have

$$4a^2 + 9b^2 - 1 = 4a^2 - 1 = (2a - 1)(2a + 1),$$

which is divisible by n.

Second, we handle the case when n is not divisible by 3. In that case, let a be 0 and let b be any integer satisfying $3b \equiv 1 \bmod n$. Then we have

$$4a^2 + 9b^2 - 1 = 9b^2 - 1 = (3b - 1)(3b + 1),$$

which is divisible by n.

Finally, we handle the general case. We can express n as $n_1 n_2$, where n_1 is not divisible by 2, n_2 is not divisible by 3, and n_1 and n_2 are relatively prime. (For example, let n_2 be the power of 2 in the prime factorization of n, and let n_1 be the rest of the prime factorization of n.) From the first paragraph, there are integers a_1 and b_1 such that $4a_1^2 + 9b_1^2 - 1$ is divisible by n_1. From the second paragraph, there are integers a_2 and b_2 such that $4a_2^2 + 9b_2^2 - 1$ is divisible by n_2.

By the Chinese Remainder Theorem, there is an integer a that is $a_1 \bmod n_1$ and $a_2 \bmod n_2$. Similarly, there is an integer b that is $b_1 \bmod n_1$ and $b_2 \bmod n_2$. Modulo n_1, we have

$$4a^2 + 9b^2 - 1 \equiv 4a_1^2 + 9b_1^2 - 1 \equiv 0 \pmod{n_1}.$$

Modulo n_2, we have

$$4a^2 + 9b^2 - 1 \equiv 4a_2^2 + 9b_2^2 - 1 \equiv 0 \pmod{n_2}.$$

Thus $4a^2 + 9b^2 - 1$ is divisible by both n_1 and n_2. Because n_1 and n_2 are relatively prime, that means $4a^2 + 9b^2 - 1$ is divisible by their product $n_1 n_2$, which is n. That's what we wanted to show.

Problem 3 Let p and q be integers such that q is nonzero. Prove that

$$\left| \frac{p}{q} - \sqrt{7} \right| \geq \frac{24 - 9\sqrt{7}}{q^2}.$$

Solution: First, we handle the case when $|p| \le \frac{8}{3}|q|$. Because $\sqrt{7}$ is irrational, we have

$$|p - q\sqrt{7}| \cdot |p + q\sqrt{7}| = |p^2 - 7q^2| \ge 1.$$

Hence we have

$$\begin{aligned} |p - q\sqrt{7}| &\ge \frac{1}{|p + q\sqrt{7}|} \\ &\ge \frac{1}{|p| + |q|\sqrt{7}} \\ &\ge \frac{1}{\frac{8}{3}|q| + |q|\sqrt{7}} \\ &= \frac{3}{(8 + 3\sqrt{7})|q|} \\ &= \frac{3(8 - 3\sqrt{7})}{|q|} \\ &= \frac{24 - 9\sqrt{7}}{|q|}. \end{aligned}$$

Dividing both sides by $|q|$ gives the desired inequality.

Second, we handle the case when $|p| \ge \frac{8}{3}|q|$. If $|q| \ge 3$, then we have

$$\begin{aligned} \left|\frac{p}{q} - \sqrt{7}\right| &\ge \left|\frac{p}{q}\right| - \sqrt{7} \\ &\ge \frac{8}{3} - \sqrt{7} \\ &= \frac{8 - 3\sqrt{7}}{3} \\ &= \frac{24 - 9\sqrt{7}}{9} \\ &\ge \frac{24 - 9\sqrt{7}}{q^2}. \end{aligned}$$

If $|q| \le 2$, then $|p| \ge \frac{8}{3}|q|$ implies $|p| \ge 3|q|$, and so

$$\begin{aligned} \left|\frac{p}{q} - \sqrt{7}\right| &\ge \left|\frac{p}{q}\right| - \sqrt{7} \\ &\ge 3 - \sqrt{7} \\ &> 24 - 9\sqrt{7} \\ &\ge \frac{24 - 9\sqrt{7}}{q^2}. \end{aligned}$$

In all cases, we have proved the desired inequality.

Problem 4 Let S be a set of n points in the coordinate plane. Say that a pair of points is *aligned* if the two points have the same x-coordinate or y-coordinate. Prove that S can be partitioned into disjoint subsets such that (a) each of these subsets is a collinear set of points, and (b) at most $n^{3/2}$ unordered pairs of distinct points in S are aligned but not in the same subset.

Solution: We will prove the result by strong induction on n. The case $n = 0$ is trivial, so from now on assume that n is positive.

Assume that the result is true for every number less than n. Let L be the largest subset of S whose points all have the same x-coordinate or all have the same y-coordinate. Without loss of generality, assume that the points in L all have the same x-coordinate. Let k be the number of points in L.

Because L is largest, for every point in L, there are at most k points in $S - L$ with the same y-coordinate as the original point. So the number of points in $S - L$ that share a y-coordinate with some point in L is at most $k \cdot k = k^2$. On the other hand, the number of such points is at most n. Since the number of such points is at most k^2 and at most n, it is at most the geometric mean of k^2 and n, which is $k\sqrt{n}$. To summarize, we have shown that the number of points in $S - L$ with the same y-coordinate as some point in L is at most $k\sqrt{n}$.

By induction applied to the set $S - L$, we can partition $S - L$ into collinear subsets so that at most $(n-k)^{3/2}$ pairs of points are aligned but not in the same subset. By including L, we get a partition of S. Now, the number of "bad" pairs of points is at most $(n-k)^{3/2} + k\sqrt{n}$. But we have

$$(n-k)^{3/2} + k\sqrt{n} = (n-k)\sqrt{n-k} + k\sqrt{n} \le (n-k)\sqrt{n} + k\sqrt{n} = n^{3/2}.$$

So we have established the result by induction.