Advantage Testing Foundation

# The Second Annual Math Prize for Girls

Saturday, November 13, 2010

## TEST BOOKLET

Test Version A

### DIRECTIONS

1. Do not open this test until your proctor instructs you to.
2. Fill out the top of your answer sheet. Your student number is on your name tag. Your test version (A, B, C, or D) is above on this page.
3. You may use pencils, pens, and erasers. You may also use the scratch paper that we provide. You may NOT use calculators, rulers, protractors, compasses, graph paper, books, or slide rules.
4. Write your final answers on the answer sheet. When a problem specifies a particular form for the answer, write your answer in that form. Write your answers clearly. We will collect only the answer sheet from you.

### NOTES

1. This test contains 20 problems. You will have 150 minutes (2.5 hours) to take the test. Your score will be the number of correct answers.
2. Figures are not necessarily drawn to scale.
3. Good luck!

# Math Prize for Girls 2010

**1.** If $a$ and $b$ are nonzero real numbers such that $|a| \neq |b|$, compute the value of the expression

$$\left(\frac{b^2}{a^2} + \frac{a^2}{b^2} - 2\right) \times \left(\frac{a+b}{b-a} + \frac{b-a}{a+b}\right) \times \left(\frac{\frac{1}{a^2} + \frac{1}{b^2}}{\frac{1}{b^2} - \frac{1}{a^2}} - \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{a^2} + \frac{1}{b^2}}\right).$$

**2.** Jane has two bags $X$ and $Y$. Bag $X$ contains 4 red marbles and 5 blue marbles (and nothing else). Bag $Y$ contains 7 red marbles and 6 blue marbles (and nothing else). Jane will choose one of her bags at random (each bag being equally likely). From her chosen bag, she will then select one of the marbles at random (each marble in that bag being equally likely). What is the probability that she will select a red marble? Express your answer as a fraction in simplest form.

**3.** How many ordered triples of integers $(x, y, z)$ are there such that

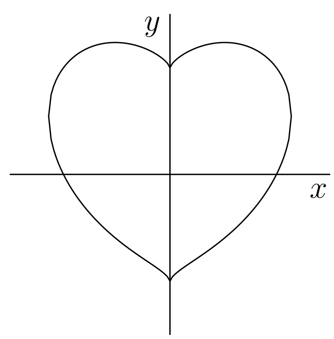
$$x^2 + y^2 + z^2 = 34\,?$$

**4.** Consider the sequence of six real numbers 60, 10, 100, 150, 30, and $x$. The average (arithmetic mean) of this sequence is equal to the median of the sequence. What is the sum of all the possible values of $x$? (The median of a sequence of six real numbers is the average of the two middle numbers after all the numbers have been arranged in increasing order.)

**5.** Find the smallest two-digit positive integer that is a divisor of 201020112012.

**6.** The bases of a trapezoid have lengths 10 and 21, and the legs have lengths $\sqrt{34}$ and $3\sqrt{5}$. What is the area of the trapezoid? Express your answer as a fraction in simplest form.

**7.** The graph of $(x^2 + y^2 - 1)^3 = x^2 y^3$ is a heart-shaped curve, shown in the figure below.



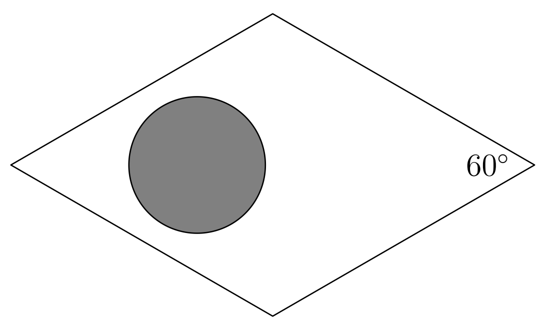

For how many ordered pairs of integers $(x, y)$ is the point $(x, y)$ inside or on this curve?

**8.** When Meena turned 16 years old, her parents gave her a cake with $n$ candles, where $n$ has exactly 16 different positive integer divisors. What is the smallest possible value of $n$?

**9.** Lynnelle took 10 tests in her math class at Stanford. Her score on each test was an integer from 0 through 100. She noticed that, for every four consecutive tests, her average score on those four tests was at most 47.5. What is the largest possible average score she could have on all 10 tests?

**10.** The triangle $ABC$ lies on the coordinate plane. The midpoint of $\overline{AB}$ has coordinates $(-16, -63)$, the midpoint of $\overline{AC}$ has coordinates $(13, 50)$, and the midpoint of $\overline{BC}$ has coordinates $(6, -85)$. What are the coordinates of point $A$? Express your answer as an ordered pair $(x, y)$.

**11.** In the figure below, each side of the rhombus has length 5 centimeters.




The circle lies entirely within the rhombus. The area of the circle is $n$ square centimeters, where $n$ is a positive integer. Compute the number of possible values of $n$.

**12.** Say that an ordered triple $(a, b, c)$ is *pleasing* if

(a) $a$, $b$, and $c$ are in the set $\{1, 2, \ldots, 17\}$, and

(b) both $b - a$ and $c - b$ are greater than 3, and at least one of them is equal to 4.

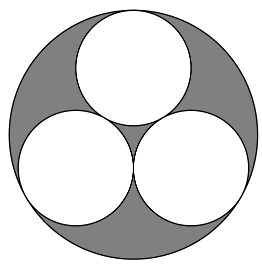
How many pleasing triples are there?

**13.** For every positive integer $n$, define $S_n$ to be the sum

$$S_n = \sum_{k=1}^{2010} \left( \cos \frac{k!\,\pi}{2010} \right)^n .$$

As $n$ approaches infinity, what value does $S_n$ approach?

**14.** In the figure below, the three small circles are congruent and tangent to each other. The large circle is tangent to the three small circles.



The area of the large circle is 1. What is the area of the shaded region? Express your answer in the form $a\sqrt{n} - b$, where $a$ and $b$ are positive integers and $n$ is a square-free positive integer.

**15.** Compute the value of the sum

$$\frac{1}{1+\tan^3 0^\circ} + \frac{1}{1+\tan^3 10^\circ} + \frac{1}{1+\tan^3 20^\circ} + \frac{1}{1+\tan^3 30^\circ} + \frac{1}{1+\tan^3 40^\circ} + \frac{1}{1+\tan^3 50^\circ} + \frac{1}{1+\tan^3 60^\circ} + \frac{1}{1+\tan^3 70^\circ} + \frac{1}{1+\tan^3 80^\circ} .$$

**16.** Let $P$ be the quadratic function such that $P(0) = 7$, $P(1) = 10$, and $P(2) = 25$. If $a$, $b$, and $c$ are integers such that every positive number $x$ less than 1 satisfies

$$\sum_{n=0}^{\infty} P(n) x^n = \frac{ax^2 + bx + c}{(1-x)^3},$$

compute the ordered triple $(a, b, c)$.

**17.** For every $x \geq -\frac{1}{e}$, there is a unique number $W(x) \geq -1$ such that

$$W(x)e^{W(x)} = x.$$

The function $W$ is called Lambert's $W$ function. Let $y$ be the unique positive number such that

$$\frac{y}{\log_2 y} = -\frac{3}{5}\,.$$

The value of $y$ is of the form $e^{-W(z \ln 2)}$ for some rational number $z$. What is the value of $z$? Express your answer as a fraction in simplest form.

**18.** If $a$ and $b$ are positive integers such that

$$\sqrt{8 + \sqrt{32 + \sqrt{768}}} = a \cos \frac{\pi}{b}\,,$$

compute the ordered pair $(a, b)$.

**19.** Let $S$ be the set of 81 points $(x, y)$ such that $x$ and $y$ are integers from $-4$ through 4. Let $A$, $B$, and $C$ be random points chosen independently from $S$, with each of the 81 points being equally likely. (The points $A$, $B$, and $C$ do not have to be different.) Let $K$ be the area of the (possibly degenerate) triangle $ABC$. What is the expected value (average value) of $K^2$? Express your answer as a fraction in simplest form.

**20.** What is the value of the sum

$$\sum_z \frac{1}{|1 - z|^2}\,,$$

where $z$ ranges over all 7 solutions (real and nonreal) of the equation $z^7 = -1$? Express your answer as a fraction in simplest form.