



MATH PRIZE *for* GIRLS

2010 SOLUTIONS

Problem 1 If a and b are nonzero real numbers such that $|a| \neq |b|$, compute the value of the expression

$$\left(\frac{b^2}{a^2} + \frac{a^2}{b^2} - 2\right) \times \left(\frac{a+b}{b-a} + \frac{b-a}{a+b}\right) \times \left(\frac{\frac{1}{a^2} + \frac{1}{b^2}}{\frac{1}{b^2} - \frac{1}{a^2}} - \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{a^2} + \frac{1}{b^2}}\right).$$

Answer: -8 .

Solution: Let's simplify the monstrous expression one step at a time:

$$\begin{aligned} & \left(\frac{b^2}{a^2} + \frac{a^2}{b^2} - 2\right) \times \left(\frac{a+b}{b-a} + \frac{b-a}{a+b}\right) \times \left(\frac{\frac{1}{a^2} + \frac{1}{b^2}}{\frac{1}{b^2} - \frac{1}{a^2}} - \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{a^2} + \frac{1}{b^2}}\right) \\ &= \frac{b^4 + a^4 - 2a^2b^2}{a^2b^2} \times \frac{(a+b)^2 + (b-a)^2}{(b-a)(a+b)} \times \left(\frac{b^2 + a^2}{a^2 - b^2} - \frac{a^2 - b^2}{b^2 + a^2}\right) \\ &= \frac{(b^2 - a^2)^2}{a^2b^2} \times \frac{2(a^2 + b^2)}{b^2 - a^2} \times \frac{(b^2 + a^2)^2 - (a^2 - b^2)^2}{(a^2 - b^2)(b^2 + a^2)} \\ &= \frac{2(b^2 - a^2)(a^2 + b^2)}{a^2b^2} \times \frac{4a^2b^2}{(a^2 - b^2)(b^2 + a^2)} \\ &= \frac{8(b^2 - a^2)}{a^2 - b^2} \\ &= \boxed{-8}. \end{aligned}$$

Problem 2 Jane has two bags X and Y . Bag X contains 4 red marbles and 5 blue marbles (and nothing else). Bag Y contains 7 red marbles and 6 blue marbles (and nothing else). Jane will choose one of her bags at random (each bag being equally likely). From her chosen bag, she will then select one of the marbles at random (each marble in that bag being equally likely). What is the probability that she will select a red marble? Express your answer as a fraction in simplest form.

Answer: $\frac{115}{234}$.

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Solution: The probability that Jane will choose bag X and then select a red marble is

$$\Pr(X \text{ and red}) = \frac{1}{2} \times \frac{4}{9} = \frac{2}{9}.$$

The probability that Jane will choose bag Y and then select a red marble is

$$\Pr(Y \text{ and red}) = \frac{1}{2} \times \frac{7}{13} = \frac{7}{26}.$$

So the probability that Jane will select a red marble is the sum

$$\begin{aligned} \Pr(\text{red}) &= \Pr(X \text{ and red}) + \Pr(Y \text{ and red}) \\ &= \frac{2}{9} + \frac{7}{26} \\ &= \frac{2(26) + 9(7)}{9(26)} \\ &= \frac{52 + 63}{234} \\ &= \boxed{\frac{115}{234}}. \end{aligned}$$

Problem 3 How many ordered triples of integers (x, y, z) are there such that

$$x^2 + y^2 + z^2 = 34?$$

Answer: 48.

Solution: We have to represent 34 as the sum of three squares chosen from 0, 1, 4, 9, 16, and 25. By playing a bit, we see that the only such sums (ignoring order) are $25 + 9 + 0$ and $16 + 9 + 9$. Let's consider the two cases separately.

Case $25 + 9 + 0$: Since x^2 , y^2 , and z^2 are 25, 9, and 0 (in some order), x , y , and z are ± 5 , ± 3 , and 0 (in some order). There are $3! = 6$ ways to choose the order, and $2^2 = 4$ ways to choose the \pm signs. So the number of such triples is $6 \cdot 4$, or 24.

Case $16 + 9 + 9$: Since x^2 , y^2 , and z^2 are 16, 9, and 9 (in some order), x , y , and z are ± 4 , ± 3 , and ± 3 (in some order). There are 3 places for ± 4 ,

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and $2^3 = 8$ ways to choose the \pm signs. So the number of such triples is $3 \cdot 8$, or 24.

There are 24 triples in the first case and 24 triples in the second case, so the total number of triples is $\boxed{48}$.

Problem 4 Consider the sequence of six real numbers 60, 10, 100, 150, 30, and x . The average (arithmetic mean) of this sequence is equal to the median of the sequence. What is the sum of all the possible values of x ? (The median of a sequence of six real numbers is the average of the two middle numbers after all the numbers have been arranged in increasing order.)

Answer: 135.

Solution: Excluding x , the sum of the five numbers is 350. So the sum of all six numbers is $x + 350$. Hence their mean is $\frac{x+350}{6}$.

Excluding x , the five numbers in order are 10, 30, 60, 100, and 150. The median of these five numbers is 60. Including x , the median will depend on the value of x .

Case $x \leq 30$: In this case, the median is $\frac{30+60}{2} = 45$. So we get the equation $\frac{x+350}{6} = 45$. The solution is $x = -80$. Since -80 is less than 30, it is a possible value of x .

Case $30 < x < 100$: In this case, the median is $\frac{x+60}{2}$. So we get the equation $\frac{x+350}{6} = \frac{x+60}{2}$. The solution is $x = 85$. Since 85 is between 30 and 100, it is another possible value of x .

Case $x > 100$: In this case, the median is $\frac{60+100}{2} = 80$. So we get the equation $\frac{x+350}{6} = 80$. The solution is $x = 130$. Since 130 is bigger than 100, it is a third possible value of x .

Combining all three cases, we see that the three possible values of x are -80 , 85, and 130. Their sum is $\boxed{135}$.

Problem 5 Find the smallest two-digit positive integer that is a divisor of 201020112012.

Answer: 12.

Solution: Let $N = 201020112012$. We're looking for a small two-digit divisor of N , so let's start from 10 and work up.

Is 10 a divisor of N ? No, because N doesn't end with a 0.

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Is 11 a divisor of N ? Let's use the divisibility rule for 11, which involves the alternating sum of the digits of N :

$$2 - 0 + 1 - 0 + 2 - 0 + 1 - 1 + 2 - 0 + 1 - 2 = 9 - 3 = 6.$$

Because 6 is not divisible by 11, neither is N .

Is 12 a divisor of N ? We have to check whether N is divisible by 3 and 4. It is divisible by 4, because its last two digits are 12, which is divisible by 4. To check whether N is divisible by 3, let's compute the sum of the digits of N :

$$2 + 0 + 1 + 0 + 2 + 0 + 1 + 1 + 2 + 0 + 1 + 2 = 12.$$

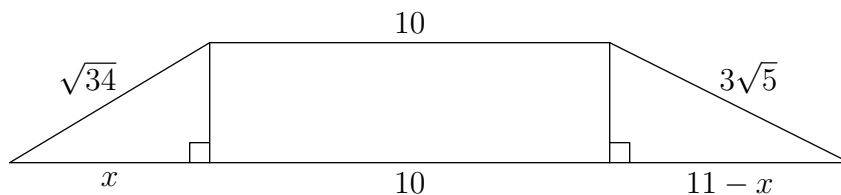
Because this sum 12 is divisible by 3, so is N . Since N is divisible by both 3 and 4, it is divisible by 12.

Hence the smallest two-digit divisor of N is 12.

Problem 6 The bases of a trapezoid have lengths 10 and 21, and the legs have lengths $\sqrt{34}$ and $3\sqrt{5}$. What is the area of the trapezoid? Express your answer as a fraction in simplest form.

Answer: $\frac{93}{2}$.

Solution: Below is a drawing of the trapezoid and two of its altitudes.



By the Pythagorean theorem applied to the left triangle, the height of the trapezoid is $\sqrt{34 - x^2}$. By Pythagoras applied to the right triangle, the height is $\sqrt{45 - (11 - x)^2}$. So we get the equation

$$34 - x^2 = 45 - (11 - x)^2 = 45 - (121 - 22x + x^2) = -76 + 22x - x^2.$$

The quadratic terms cancel. Solving for x gives

$$x = \frac{34 + 76}{22} = \frac{110}{22} = 5.$$

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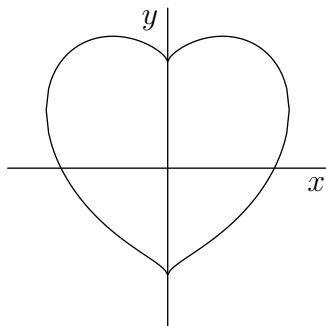
As a result, the height of the trapezoid is

$$\sqrt{34 - x^2} = \sqrt{34 - 5^2} = \sqrt{34 - 25} = \sqrt{9} = 3.$$

So the area of the trapezoid is

$$3 \times \frac{10 + 21}{2} = 3 \times \frac{31}{2} = \boxed{\frac{93}{2}}.$$

Problem 7 The graph of $(x^2 + y^2 - 1)^3 = x^2y^3$ is a heart-shaped curve, shown in the figure below.

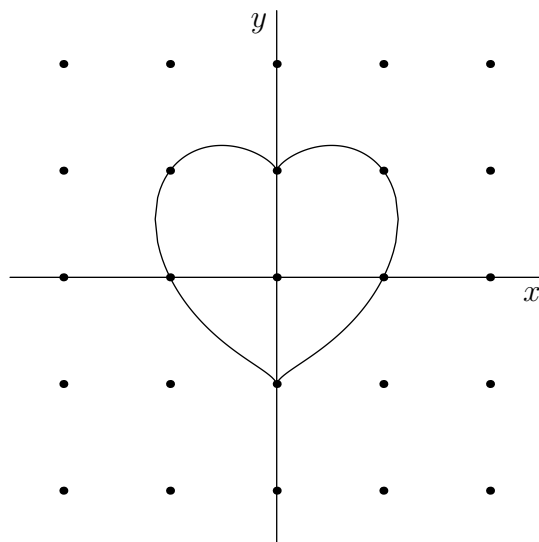


For how many ordered pairs of integers (x, y) is the point (x, y) inside or on this curve?

Answer: 7.

Solution: The equation of the heart is $(x^2 + y^2 - 1)^3 = x^2y^3$. Plugging in $y = 0$, the equation becomes $x^2 - 1 = 0$, so $x = \pm 1$. Plugging in $y = 1$, the equation becomes $(x^2)^3 = x^2$, so x is $-1, 0$, or 1 . Plugging in $y = -1$, the equation becomes $(x^2)^3 = -x^2$, so $x = 0$.

We have identified 6 integer points on the heart, as shown in the picture below.



The points inside the heart satisfy $(x^2 + y^2 - 1)^3 < x^2 y^3$. So the origin $(0, 0)$ is inside the heart. Looking at the picture, we see that every point in or on the heart satisfies $|y| < 2$. So we have found all the integer points in or on the heart. Hence the number of such points is $6 + 1 = \boxed{7}$.

Problem 8 When Meena turned 16 years old, her parents gave her a cake with n candles, where n has exactly 16 different positive integer divisors. What is the smallest possible value of n ?

Answer: 120.

Solution: To have 16 positive divisors, n must be of the form p^{15} , $p^7 q$, $p^3 q^3$, $p^3 q r$, or $p q r s$, where p , q , r , and s are different primes. Let's find the smallest n in each of these five cases.

Case $n = p^{15}$: In this case, the smallest n is 2^{15} , or 32,768.

Case $n = p^7 q$: In this case, the smallest n is $2^7 \cdot 3$, or 384.

Case $n = p^3 q^3$: In this case, the smallest n is $2^3 \cdot 3^3$, or 216.

Case $n = p^3 q r$: In this case, the smallest n is $2^3 \cdot 3 \cdot 5$, or 120.

Case $n = p q r s$: In this case, the smallest n is $2 \cdot 3 \cdot 5 \cdot 7$, or 210.

Looking at all five cases, we see that the smallest possible value of n is $\boxed{120}$.

Problem 9 Lynnelle took 10 tests in her math class at Stanford. Her score on each test was an integer from 0 through 100. She noticed that, for every

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four consecutive tests, her average score on those four tests was at most 47.5. What is the largest possible average score she could have on all 10 tests?

Answer: 57.

Solution: Let the 10 scores be s_1, s_2, \dots, s_{10} . For each four consecutive scores, their average is at most 47.5, so their sum is at most 4×47.5 , which is 190. In particular, we have

$$\begin{aligned}s_1 + s_2 + s_3 + s_4 &\leq 190 \\s_4 + s_5 + s_6 + s_7 &\leq 190 \\s_7 + s_8 + s_9 + s_{10} &\leq 190.\end{aligned}$$

Adding all three inequalities, we get

$$s_1 + s_2 + s_3 + 2s_4 + s_5 + s_6 + 2s_7 + s_8 + s_9 + s_{10} \leq 570.$$

So the sum of all 10 scores is at most 570. Hence the average of all 10 scores is at most 57.

Let's show that an average score of 57 is possible. Consider the 10 scores

$$95, 95, 0, 0, 95, 95, 0, 0, 95, 95.$$

Each score is an integer from 0 through 100. For each four consecutive scores, their sum is 190, and so their average is 47.5. The sum of all 10 scores is 570, and so their average is 57. Hence an average score of 57 is possible.

Combining our observations, we see that the largest possible average score is $\boxed{57}$.

Problem 10 The triangle ABC lies on the coordinate plane. The midpoint of \overline{AB} has coordinates $(-16, -63)$, the midpoint of \overline{AC} has coordinates $(13, 50)$, and the midpoint of \overline{BC} has coordinates $(6, -85)$. What are the coordinates of point A ? Express your answer as an ordered pair (x, y) .

Answer: $(-9, 72)$.

Solution: Because the midpoint of \overline{AB} is $(-16, -63)$, we get the equation

$$\frac{A + B}{2} = (-16, 63).$$

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Similarly, we get the equations

$$\frac{A + C}{2} = (13, 50)$$

and

$$\frac{B + C}{2} = (6, -85).$$

Adding the first two equations gives

$$A + \frac{B + C}{2} = (-3, -13).$$

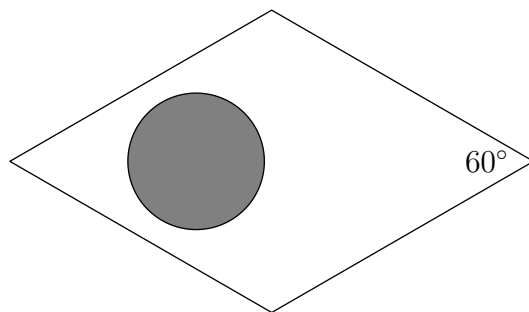
Substituting the third equation into the last equation gives

$$A + (6, -85) = (-3, -13).$$

Solving for A gives

$$A = (-6, 85) + (-3, -13) = \boxed{(-9, 72)}.$$

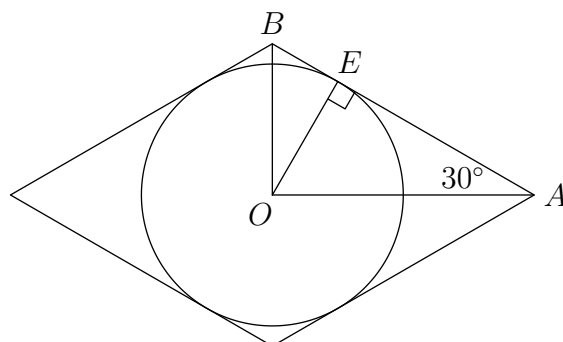
Problem 11 In the figure below, each side of the rhombus has length 5 centimeters.



The circle lies entirely within the rhombus. The area of the circle is n square centimeters, where n is a positive integer. Compute the number of possible values of n .

Answer: 14.

Solution: Let's consider the case when the circle and rhombus have the same center and are tangent to each other. See the following picture.



Because triangle ABO is a 30-60-90 triangle with $AB = 5$ cm, we have $OA = \frac{5\sqrt{3}}{2}$ cm. Because triangle AEO is a 30-60-90 triangle, we have $OE = \frac{5\sqrt{3}}{4}$ cm. In other words, the radius of the circle (in centimeters) is $r = \frac{5\sqrt{3}}{4}$. Thus the area of the circle (in square centimeters) is

$$A = \pi r^2 = \pi \left(\frac{5}{4}\sqrt{3} \right)^2 = \pi \frac{25}{16}(3) = \frac{75\pi}{16}.$$

Because $\pi < 3.2$, the area satisfies the upper bound

$$A = \frac{75\pi}{16} < \frac{75(3.2)}{16} = 75(0.2) = 15.$$

Because $\pi > 3$, the area satisfies the lower bound

$$A = \frac{75\pi}{16} > \frac{75(3)}{16} = \frac{225}{16} > 14.$$

So the area of this big circle is between 14 and 15.

It's clear that no circle inside the rhombus can be larger than the circle above. By shrinking our big circle, we can have circles of area 1, 2, \dots , 14. So the number of possible integer areas is 14.

Problem 12 Say that an ordered triple (a, b, c) is *pleasing* if

1. a , b , and c are in the set $\{1, 2, \dots, 17\}$, and
2. both $b - a$ and $c - b$ are greater than 3, and at least one of them is equal to 4.

How many pleasing triples are there?

Answer: 81.

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Solution: The value of b can be 5, 6, \dots , 13. Let's fix one of these 9 values of b . The possible values of a split into two cases.

Case $a \leq b - 5$: Then c must be $b + 4$. Since there are $b - 5$ possible values of a , there are $b - 5$ pleasing triples in this case.

Case $a = b - 4$: Then c could be $b + 4, b + 5, \dots, 17$. There are $14 - b$ possible values of c , so there are $14 - b$ pleasing triples in this case.

Combining both cases, we see that for each fixed b , the number of pleasing triples is

$$(b - 5) + (14 - b) = 9.$$

Since there are 9 possible values of b , the total number of pleasing triples is $9 \cdot 9$, or 81.

Problem 13 For every positive integer n , define S_n to be the sum

$$S_n = \sum_{k=1}^{2010} \left(\cos \frac{k! \pi}{2010} \right)^n.$$

As n approaches infinity, what value does S_n approach?

Answer: 1944.

Solution: The prime factorization of 2010 is $2 \cdot 3 \cdot 5 \cdot 67$. So let's split the sum into two cases, depending on how the index k compares with 67.

Case $k \leq 66$: In this case, $k!$ is not a multiple of 67, and so is not a multiple of 2010. Hence $\frac{k! \pi}{2010}$ is not a multiple of π . Thus $\cos \frac{k! \pi}{2010}$ is strictly between -1 and 1 . Therefore as n approaches infinity, $(\cos \frac{k! \pi}{2010})^n$ approaches zero.

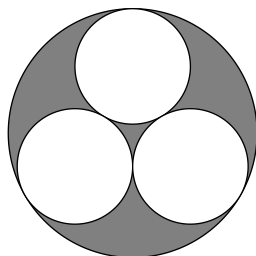
Case $k \geq 67$: In this case, k is a multiple of $2^2, 3, 5$, and 67 , and so is a multiple of $2 \cdot 2010$. Hence $\frac{k! \pi}{2010}$ is a multiple of 2π . Thus $\cos \frac{k! \pi}{2010}$ is 1. Therefore $(\cos \frac{k! \pi}{2010})^n$ is always 1.

Combining the two cases, we see that each $k \leq 66$ contributes nothing to the limit, while each $k \geq 67$ contributes 1. The number of k from 67 to 2010 is

$$2010 - 67 + 1 = 1943 + 1 = 1944.$$

So the sum approaches 1944.

Problem 14 In the figure below, the three small circles are congruent and tangent to each other. The large circle is tangent to the three small circles.

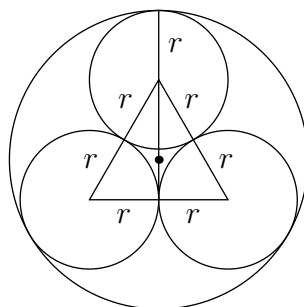


The area of the large circle is 1. What is the area of the shaded region? Express your answer in the form $a\sqrt{n} - b$, where a and b are positive integers and n is a square-free positive integer.

Answer: $36\sqrt{3} - 62$.

Solution: Let R be the radius of the large circle. Because the area of the large circle is 1, we have $\pi R^2 = 1$.

Let r be the radius of the three small circles. Consider the equilateral triangle whose vertices are the centers of the three small circles, as shown below.



Each side of the equilateral triangle has length $2r$. By properties of 30-60-90 triangles, each median of this triangle has length $r\sqrt{3}$. Because the centroid of the triangle divides each median in a $2 : 1$ ratio, we have $R - r = \frac{2}{3}r\sqrt{3}$. Clearing fractions gives $3R - 3r = 2r\sqrt{3}$. Solving for r gives

$$r = \frac{3}{2\sqrt{3} + 3}R = (2\sqrt{3} - 3)R.$$

So the area of each small circle is

$$\pi r^2 = \pi(2\sqrt{3} - 3)^2 R^2 = (2\sqrt{3} - 3)^2 = 12 - 12\sqrt{3} + 9 = 21 - 12\sqrt{3}.$$

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The combined area of all three small circles is thus

$$3(21 - 12\sqrt{3}) = 63 - 36\sqrt{3}.$$

Hence the area of the shaded region is

$$1 - (63 - 36\sqrt{3}) = \boxed{36\sqrt{3} - 62}.$$

Problem 15 Compute the value of the sum

$$\begin{aligned} & \frac{1}{1 + \tan^3 0^\circ} + \frac{1}{1 + \tan^3 10^\circ} + \frac{1}{1 + \tan^3 20^\circ} + \frac{1}{1 + \tan^3 30^\circ} + \frac{1}{1 + \tan^3 40^\circ} \\ & \quad + \frac{1}{1 + \tan^3 50^\circ} + \frac{1}{1 + \tan^3 60^\circ} + \frac{1}{1 + \tan^3 70^\circ} + \frac{1}{1 + \tan^3 80^\circ}. \end{aligned}$$

Answer: 5.

Solution: The first term is

$$\frac{1}{1 + \tan^3 0^\circ} = \frac{1}{1 + 0^3} = \frac{1}{1 + 0} = 1.$$

Because 10° and 80° are complements, $\tan 10^\circ$ and $\tan 80^\circ$ are reciprocals. So the second and ninth terms add up to

$$\begin{aligned} \frac{1}{1 + \tan^3 10^\circ} + \frac{1}{1 + \tan^3 80^\circ} &= \frac{1 + \tan^3 80^\circ + 1 + \tan^3 10^\circ}{(1 + \tan^3 10^\circ)(1 + \tan^3 80^\circ)} \\ &= \frac{2 + \tan^3 10^\circ + \tan^3 80^\circ}{1 + \tan^3 10^\circ + \tan^3 80^\circ + \tan^3 10^\circ \tan^3 80^\circ} \\ &= \frac{2 + \tan^3 10^\circ + \tan^3 80^\circ}{1 + \tan^3 10^\circ + \tan^3 80^\circ + 1} \\ &= 1. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \frac{1}{1 + \tan^3 20^\circ} + \frac{1}{1 + \tan^3 70^\circ} &= 1 \\ \frac{1}{1 + \tan^3 30^\circ} + \frac{1}{1 + \tan^3 60^\circ} &= 1 \\ \frac{1}{1 + \tan^3 40^\circ} + \frac{1}{1 + \tan^3 50^\circ} &= 1. \end{aligned}$$

Adding all five displayed equations, we see that the entire sum is $\boxed{5}$.

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Problem 16 Let P be the quadratic function such that $P(0) = 7$, $P(1) = 10$, and $P(2) = 25$. If a , b , and c are integers such that every positive number x less than 1 satisfies

$$\sum_{n=0}^{\infty} P(n)x^n = \frac{ax^2 + bx + c}{(1-x)^3},$$

compute the ordered triple (a, b, c) .

Answer: $(16, -11, 7)$.

Solution: From the given initial conditions, we have

$$\sum_{n=0}^{\infty} P(n)x^n = 7 + 10x + 25x^2 + \dots$$

Multiplying by $1 - x$ gives

$$(1-x) \sum_{n=0}^{\infty} P(n)x^n = 7 + 3 + 15x^2 + \dots$$

Multiplying by $1 - x$ again gives

$$(1-x)^2 \sum_{n=0}^{\infty} P(n)x^n = 7 - 4x + 12x^2 + \dots$$

Multiplying by $1 - x$ a third time gives

$$(1-x)^3 \sum_{n=0}^{\infty} P(n)x^n = 7 - 11x + 16x^2 + \dots$$

Comparing coefficients, we see that $a = 16$, $b = -11$, and $c = 7$. So the desired triple is $\boxed{(16, -11, 7)}$.

Problem 17 For every $x \geq -\frac{1}{e}$, there is a unique number $W(x) \geq -1$ such that

$$W(x)e^{W(x)} = x.$$

The function W is called Lambert's W function. Let y be the unique positive number such that

$$\frac{y}{\log_2 y} = -\frac{3}{5}.$$

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The value of y is of the form $e^{-W(z \ln 2)}$ for some rational number z . What is the value of z ? Express your answer as a fraction in simplest form.

Answer: $\frac{5}{3}$.

Solution: Replacing y with the expression involving z , we have

$$\begin{aligned} -\frac{3}{5} &= \frac{y}{\log_2 y} \\ &= \frac{e^{-W(z \ln 2)}}{\log_2 e^{-W(z \ln 2)}} \\ &= \frac{e^{-W(z \ln 2)}}{-W(z \ln 2) \log_2 e} \\ &= -\frac{1}{W(z \ln 2) e^{W(z \ln 2)} \log_2 e} \\ &= -\frac{1}{z \ln(2) \log_2(e)} \\ &= -\frac{1}{z}. \end{aligned}$$

Solving for z , we get $z = \boxed{\frac{5}{3}}$.

Problem 18 If a and b are positive integers such that

$$\sqrt{8 + \sqrt{32 + \sqrt{768}}} = a \cos \frac{\pi}{b},$$

compute the ordered pair (a, b) .

Answer: $(4, 24)$.

Solution: Let's first simplify $\sqrt{768}$:

$$\sqrt{768} = \sqrt{256 \cdot 3} = 16\sqrt{3}.$$

Since the problem asks for cosine, let's express $\sqrt{768}$ with cosine:

$$\sqrt{768} = 16\sqrt{3} = 32 \frac{\sqrt{3}}{2} = 32 \cos \frac{\pi}{6}.$$

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Next, we have

$$\sqrt{32 + \sqrt{768}} = \sqrt{32 + 32 \cos \frac{\pi}{6}} = 4\sqrt{2 + 2 \cos \frac{\pi}{6}}.$$

How can we simplify a square root of a cosine? We can use the half-angle identity

$$\cos \frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$

where x is any number between $-\pi$ and π . So we get

$$\sqrt{32 + \sqrt{768}} = 4\sqrt{2 + 2 \cos \frac{\pi}{6}} = 8\sqrt{\frac{1 + \cos \frac{\pi}{6}}{2}} = 8 \cos \frac{\pi}{12}.$$

Now let's use the half-angle identity one more time:

$$\sqrt{8 + \sqrt{32 + \sqrt{768}}} = \sqrt{8 + 8 \cos \frac{\pi}{12}} = 4\sqrt{\frac{1 + \cos \frac{\pi}{12}}{2}} = 4 \cos \frac{\pi}{24}.$$

So $a = 4$ and $b = 24$ works. By case analysis on a , we see that no other pair works. Hence the answer is $\boxed{(4, 24)}$.

Problem 19 Let S be the set of 81 points (x, y) such that x and y are integers from -4 through 4 . Let A , B , and C be random points chosen independently from S , with each of the 81 points being equally likely. (The points A , B , and C do not have to be different.) Let K be the area of the (possibly degenerate) triangle ABC . What is the expected value (average value) of K^2 ? Express your answer as a fraction in simplest form.

Answer: $\frac{200}{3}$.

Solution: Let $A = (x_1, y_1)$, let $B = (x_2, y_2)$, and let $C = (x_3, y_3)$. By the shoelace formula (also called the surveyor's area formula), the area of $\triangle ABC$ is

$$K = \frac{1}{2} |x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3|.$$

Squaring gives

$$K^2 = \frac{1}{4} (x_1^2y_2^2 + x_2^2y_3^2 + x_3^2y_1^2 + x_2^2y_1^2 + x_3^2y_2^2 + x_1^2y_3^2 + \text{cross terms}).$$

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The cross terms are terms like $x_1x_2y_2y_3$ or $x_1^2y_1y_3$; in each cross term, at least one variable is not squared.

Writing E for expected value, we shall take the expected value of both sides of the last equation:

$$E(K^2) = \frac{1}{4} \left[E(x_1^2y_2^2) + E(x_2^2y_3^2) + E(x_3^2y_1^2) + E(x_2^2y_1^2) + E(x_3^2y_2^2) + E(x_1^2y_3^2) + E(\text{cross terms}) \right].$$

Let's analyze the first term $E(x_1^2y_2^2)$. Because x_1 and y_2 are independent, we have

$$E(x_1^2y_2^2) = E(x_1^2)E(y_2^2).$$

We can compute $E(x_1^2)$ as follows:

$$\begin{aligned} E(x_1^2) &= \frac{(-4)^2 + (-3)^2 + \cdots + 4^2}{9} \\ &= \frac{2(1^2 + 2^2 + 3^2 + 4^2)}{9} \\ &= \frac{2(30)}{9} \\ &= \frac{20}{3}. \end{aligned}$$

Similarly, $E(y_2^2)$ is $\frac{20}{3}$. So $E(x_1^2y_2^2)$ is $(\frac{20}{3})^2$, which is $\frac{400}{9}$. The first 6 terms each have this value. Hence

$$\begin{aligned} E(K^2) &= \frac{1}{4} \left[6 \cdot \frac{400}{9} + E(\text{cross terms}) \right] \\ &= \frac{200}{3} + \frac{1}{4}E(\text{cross terms}). \end{aligned}$$

Let's analyze one of the cross terms like $x_1^2y_1y_3$. By independence, we have

$$E(x_1^2y_1y_3) = E(x_1^2)E(y_1)E(y_3).$$

By symmetry, $E(y_1)$ is zero, and so the whole expression is zero. In the same way, the expected value of every cross term is zero. We can thus ignore the

cross terms. The expected value of K^2 is $\boxed{\frac{200}{3}}$.

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Problem 20 What is the value of the sum

$$\sum_z \frac{1}{|1-z|^2},$$

where z ranges over all 7 solutions (real and nonreal) of the equation $z^7 = -1$? Express your answer as a fraction in simplest form.

Answer: $\frac{49}{4}$.

Solution: Let z satisfy $z^7 = -1$. By the formula for a geometric series, we have

$$1 + z + z^2 + z^3 + z^4 + z^5 + z^6 = \frac{1 - z^7}{1 - z} = \frac{1 - (-1)}{1 - z} = \frac{2}{1 - z}.$$

Dividing by 2, we have

$$\frac{1}{1 - z} = \frac{1}{2} \sum_{j=0}^6 z^j.$$

Taking the squared magnitude of both sides, we get

$$\frac{1}{|1 - z|^2} = \frac{1}{4} \left| \sum_{j=0}^6 z^j \right|^2.$$

Because $|\omega|^2 = \omega\bar{\omega}$, we have

$$\frac{1}{|1 - z|^2} = \frac{1}{4} \sum_{j=0}^6 z^j \sum_{k=0}^6 \bar{z}^k = \frac{1}{4} \sum_{j=0}^6 z^j \sum_{k=0}^6 z^{-k} = \frac{1}{4} \sum_{j=0}^6 \sum_{k=0}^6 z^{j-k}.$$

Now let's sum over all such z . We get

$$\sum_z \frac{1}{|1 - z|^2} = \frac{1}{4} \sum_{j=0}^6 \sum_{k=0}^6 \sum_z z^{j-k}.$$

If $j = k$, then we have

$$\sum_z z^{j-k} = \sum_z z^0 = \sum_z 1 = 7.$$

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If $j \neq k$, then $j - k$ is a nonzero integer between -6 and 6 . The 7 values of z are of the form $-\omega^\ell$, where $\omega = e^{2\pi i/7}$ and ℓ is an integer between 0 and 6. By the formula for a geometric series, we have

$$\sum_z z^{j-k} = (-1)^{j-k} \sum_{\ell=0}^6 \omega^{(j-k)\ell} = (-1)^{j-k} \frac{1 - \omega^{7(j-k)}}{1 - \omega^{j-k}} = 0.$$

Hence the total sum is

$$\sum_z \frac{1}{|1 - z|^2} = \frac{1}{4} \sum_{j=0}^6 7 = \frac{1}{4} \cdot 7 \cdot 7 = \boxed{\frac{49}{4}}.$$