Math Prize for Girls Olympiad

November 10, 2011

1. Let A_0, A_1, A_2, $\ldots$, A_n be nonnegative numbers such that

$$A_0 \le A_1 \le A_2 \le \cdots \le A_n.$$

Prove that

$$\left| \sum_{i=0}^{\lfloor n/2 \rfloor} A_{2i} - \frac{1}{2} \sum_{i=0}^{n} A_i \right| \le \frac{A_n}{2}.$$

(Note: $\lfloor x \rfloor$ means the greatest integer that is less than or equal to x.)

2. Let $\triangle ABC$ be an equilateral triangle. If $0 < r < 1$, let D_r be the point on $\overline{AB}$ such that $AD_r = r \cdot AB$, let E_r be the point on $\overline{BC}$ such that $BE_r = r \cdot BC$, and let P_r be the point where $\overline{AE_r}$ and $\overline{CD_r}$ intersect. Prove that the set of points P_r (over all $0 < r < 1$) lie on a circle.

3. Let n be a positive integer such that $n + 1$ is divisible by 24. Prove that the sum of all the positive divisors of n is divisible by 24.

4. Let M be a matrix with r rows and c columns. Each entry of M is a nonnegative integer. Let a be the average of all rc entries of M. If $r > (10a + 10)^c$, prove that M has two identical rows.