MATH PRIZE *for* GIRLS

2011 Olympiad Solutions

Problem 1 Let A_0, A_1, A_2, $\ldots$, A_n be nonnegative numbers such that

$$A_0 \le A_1 \le A_2 \le \cdots \le A_n.$$

Prove that

$$\left| \sum_{i=0}^{\lfloor n/2 \rfloor} A_{2i} - \frac{1}{2} \sum_{i=0}^{n} A_i \right| \le \frac{A_n}{2}.$$

(Note: $\lfloor x \rfloor$ means the greatest integer that is less than or equal to x.)

Solution: We will first handle the case when n is even. We can write our main expression as an alternating sum:

$$\begin{aligned} &\sum_{i=0}^{\lfloor n/2 \rfloor} A_{2i} - \frac{1}{2} \sum_{i=0}^{n} A_i \\ &\qquad = (A_0 + A_2 + A_4 + \cdots + A_n) - \frac{1}{2}(A_0 + A_1 + A_2 + \cdots + A_n) \\ &\qquad = \frac{1}{2}(A_0 - A_1 + A_2 - A_3 + \cdots + A_n). \end{aligned}$$

We then get the following upper bound:

$$\begin{aligned} &\sum_{i=0}^{\lfloor n/2 \rfloor} A_{2i} - \frac{1}{2} \sum_{i=0}^{n} A_i \\ &\qquad = \frac{1}{2}(A_0 - A_1 + A_2 - A_3 + \cdots + A_n) \\ &\qquad = \frac{1}{2}\Big[(A_0 - A_1) + (A_2 - A_3) + \cdots + (A_{n-2} - A_{n-1}) + A_n\Big] \\ &\qquad \le \frac{1}{2}\Big[0 + 0 + \cdots + 0 + A_n\Big] \\ &\qquad = \frac{A_n}{2}. \end{aligned}$$

We get a lower bound too:

$$\begin{aligned}
&\sum_{i=0}^{\lfloor n/2 \rfloor} A_{2i} - \frac{1}{2} \sum_{i=0}^{n} A_i \\
&\qquad = \frac{1}{2}(A_0 - A_1 + A_2 - A_3 + \cdots + A_n) \\
&\qquad = \frac{1}{2}\Big[A_0 + (-A_1 + A_2) + (-A_3 + A_4) + \cdots + (-A_{n-1} + A_n)\Big] \\
&\qquad \geq \frac{1}{2}\Big[A_0 + 0 + 0 + \cdots + 0\Big] \\
&\qquad = \frac{1}{2}A_0.
\end{aligned}$$

Our main expression is between $\frac{A_0}{2}$ and $\frac{A_n}{2}$. Hence its absolute value is at most $\frac{A_n}{2}$. We have settled the case when n is even.

Now we will handle the case when n is odd. We can again write our main expression as an alternating sum:

$$\begin{aligned}
&\sum_{i=0}^{\lfloor n/2 \rfloor} A_{2i} - \frac{1}{2} \sum_{i=0}^{n} A_i \\
&\qquad = (A_0 + A_2 + A_4 + \cdots + A_{n-1}) - \frac{1}{2}(A_0 + A_1 + A_2 + \cdots + A_n) \\
&\qquad = \frac{1}{2}(A_0 - A_1 + A_2 - A_3 + \cdots + A_{n-1} - A_n).
\end{aligned}$$

We get the following upper bound:

$$\begin{aligned}
&\sum_{i=0}^{\lfloor n/2 \rfloor} A_{2i} - \frac{1}{2} \sum_{i=0}^{n} A_i \\
&\qquad = \frac{1}{2}(A_0 - A_1 + A_2 - A_3 + \cdots + A_{n-1} - A_n) \\
&\qquad = \frac{1}{2}\Big[(A_0 - A_1) + (A_2 - A_3) + \cdots + (A_{n-1} - A_n)\Big] \\
&\qquad \leq \frac{1}{2}\Big[0 + 0 + \cdots + 0\Big] \\
&\qquad = 0.
\end{aligned}$$

We also get a lower bound:

$$\begin{aligned}
&\sum_{i=0}^{\lfloor n/2 \rfloor} A_{2i} - \frac{1}{2}\sum_{i=0}^{n} A_i \\
&\qquad = \frac{1}{2}(A_0 - A_1 + A_2 - A_3 + \cdots + A_{n-1} - A_n) \\
&\qquad = \frac{1}{2}\Big[A_0 + (-A_1 + A_2) + \cdots + (-A_{n-2} + A_{n-1}) - A_n\Big] \\
&\qquad \ge \frac{1}{2}\Big[A_0 + 0 + 0 + \cdots + 0 - A_n\Big] \\
&\qquad = \frac{1}{2}(A_0 - A_n) \\
&\qquad \ge -\frac{1}{2}A_n.
\end{aligned}$$

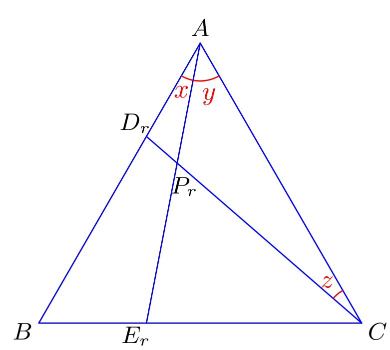
Our main expression is between $-\frac{A_n}{2}$ and 0. Hence its absolute value is at most $\frac{A_n}{2}$. We have settled the case when n is odd too.

Problem 2 Let $\triangle ABC$ be an equilateral triangle. If $0 < r < 1$, let D_r be the point on $\overline{AB}$ such that $AD_r = r \cdot AB$, let E_r be the point on $\overline{BC}$ such that $BE_r = r \cdot BC$, and let P_r be the point where $\overline{AE_r}$ and $\overline{CD_r}$ intersect. Prove that the set of points P_r (over all $0 < r < 1$) lie on a circle.

Solution: Because $\triangle ABC$ is equilateral, $AB = CA$. Because $\triangle ABC$ is equiangular, $\angle ABE_r$ and $\angle CAD_r$ are both equal to 60°. By the definitions of D_r and E_r, we have

$$BE_r = r \cdot BC = r \cdot AB = AD_r.$$

Hence, by Side-Angle-Side congruence, $\triangle ABE_r$ and $\triangle CAD_r$ are congruent.



Because $\triangle ABE_r$ is congruent to $\triangle CAD_r$, we have $\angle BAE_r = \angle ACD_r$. In other words, using the angle variables in the figure, $x = z$. Because $\angle A$ is 60°, we have $x + y = 60°$. Hence $y + z = 60°$. Because the angles of $\triangle AP_rC$ add up to 180°, we have

$$\angle AP_rC = 180° - (y + z) = 180° - 60° = 120°.$$

Consider two points P_r and P_s. By the previous paragraph

$$\angle AP_rC = 120° = \angle AP_sC.$$

Hence, because P_r and P_s are on the same side of line AC, the points A, P_r, P_s, and C lie on a circle. To be specific, consider the circle that passes through A, $P_{1/2}$, and C. Then every point P_r lies on this circle.

Note: Ken Fan proposed the original version of this problem. Ken's version asked about the locus of P_r when $\triangle ABC$ was not assumed equilateral.

Problem 3 Let n be a positive integer such that $n + 1$ is divisible by 24. Prove that the sum of all the positive divisors of n is divisible by 24.

Solution: Because $n + 1$ is divisible by 24, it is divisible by 8 (and hence 4) and by 3. So n is 3 mod 4 and 2 mod 3. Every integer squared is either 0 or 1 mod 4, so n is not a perfect square.

Let a and b be divisors of n such that $ab = n$. We claim that $(a+1)(b+1)$ is divisible by 24. To prove the claim, we will show that $(a + 1)(b + 1)$ is divisible by 3 and by 8. First, because $ab = n \equiv 2 \pmod 3$, either a or b is 2 mod 3. So either $a + 1$ or $b + 1$ is divisible by 3, which means that $(a + 1)(b + 1)$ is divisible by 3. Second, because $ab = n \equiv 3 \pmod 4$, we know that a and b are 1 and 3 mod 4 (in some order). So $a + 1$ and $b + 1$ are 2 and 0 mod 4 (in some order), which means that $(a + 1)(b + 1)$ is divisible by 8. Summarizing, we have shown that $(a + 1)(b + 1)$ is divisible by 24.

Now we show that if a and b are divisors of n such that $ab = n$, then $a + b$ is divisible by 24. We have

$$(a + 1)(b + 1) = ab + a + b + 1 = n + a + b + 1 = (n + 1) + (a + b).$$

The previous paragraph showed that $(a + 1)(b + 1)$ is divisible by 24, and by hypothesis $n + 1$ is divisible by 24. So $a + b$ is divisible by 24 too.

Now consider all the positive divisors of n. Because n is not a perfect square, these divisors can be partitioned into pairs $\{a, b\}$ such that $ab = n$.

But the previous paragraph showed that $a+b$ is divisible by 24. Since the sum of each pair is divisible by 24, the sum of all the positive divisors is divisible by 24 too.

Problem 4 Let M be a matrix with r rows and c columns. Each entry of M is a nonnegative integer. Let a be the average of all rc entries of M. If $r > (10a+10)^c$, prove that M has two identical rows.

Solution: Define the *weight* of an entry x of the matrix to be p^x, where $p = \frac{a}{a+1}$. The *weight* of a row of the matrix is the product of the weights of the entries in that row. The *weight* of the matrix is the sum of the weights of its rows.

Assume that M has no two identical rows. We will show that $r \le (10a+10)^c$. To do so, we will analyze the weight of the matrix. First, we will bound the weight from above. We claim that the weight of the matrix is at most

$$\left(1+p+p^2+\dots\right)^c.$$

Imagine expanding this expression. We would get the weight of every possible row. Because no row of M repeats, the weight of M is at most this expression. By geometric series, this expression evaluates to

$$\left(\frac{1}{1-p}\right)^c = \left(\frac{1}{1-\frac{a}{a+1}}\right)^c = (a+1)^c.$$

So the weight of the matrix is at most $(a+1)^c$.

We will now develop a lower bound on the weight of the matrix. Let S_i be the sum of the entries of row i. The weight of row i is p^{S_i}. So the weight of the matrix is

$$\sum_{i=1}^{r} p^{S_i}.$$

By the AM-GM inequality, this weight is at least

$$r\left(\prod_{i=1}^{r} p^{S_i}\right)^{1/r} = r\left(p^{\sum_{i=1}^{r} S_i}\right)^{1/r}.$$

The sum of the row sums S_i is the sum of the entries in the whole matrix, which is arc. So the weight of the matrix is at least

$$r\left(p^{arc}\right)^{1/r} = rp^{ac}.$$

Comparing the lower and upper bounds on the weight of the matrix, we get the inequality

$$rp^{ac} \leq (a+1)^c.$$

Solving for r, we find

$$r \leq \left(\frac{1}{p}\right)^{ac} (a+1)^c = \left(\frac{a+1}{a}\right)^{ac} (a+1)^c = \left(\frac{(a+1)^{a+1}}{a^a}\right)^c.$$

Now all we have to do is show that

$$\frac{(a+1)^{a+1}}{a^a} \leq 10a + 10.$$

It is well-known that $(1 + \frac{1}{a})^a$ is an increasing function of a that converges to e, the base of the natural logarithm. In particular, we have

$$\left(1 + \frac{1}{a}\right)^a < e.$$

That's equivalent to

$$\frac{(a+1)^a}{a^a} < e.$$

Multiplying by $a+1$ gives

$$\frac{(a+1)^{a+1}}{a^a} < e(a+1).$$

Because $e < 10$, we have

$$\frac{(a+1)^{a+1}}{a^a} < 10(a+1) = 10a + 10.$$

That's all we had left to show.