Advantage Testing Foundation

The Third Annual Math Prize for Girls

Saturday, September 17, 2011

TEST BOOKLET

Test Version A

DIRECTIONS

1. Do not open this test until your proctor instructs you to.
2. Fill out the top of your answer sheet. Your student number is on your name tag. Your test version (A, B, C, or D) is above on this page.
3. You may use pencils, pens, and erasers. You may also use the scratch paper that we provide. You may NOT use calculators, rulers, protractors, compasses, graph paper, books, or slide rules.
4. Write your final answers on the answer sheet. When a problem specifies a particular form for the answer, write your answer in that form. Write your answers clearly. We will collect only the answer sheet from you.

NOTES

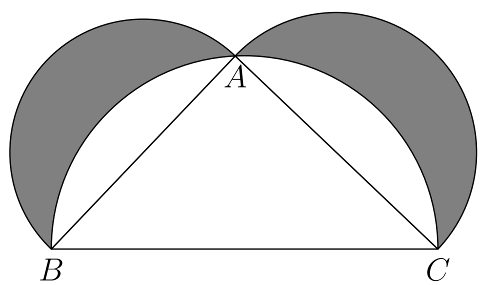
1. This test contains 20 problems. You will have 150 minutes (2.5 hours) to take the test. Your score will be the number of correct answers.
2. Figures are not necessarily drawn to scale.
3. Good luck!

Math Prize for Girls 2011

1. If m and n are integers such that $3m + 4n = 100$, what is the smallest possible value of $|m - n|$?

2. Express $\sqrt{2 + \sqrt{3}}$ in the form $\dfrac{a + \sqrt{b}}{\sqrt{c}}$, where a is a positive integer and b and c are square-free positive integers.

3. The figure below shows a triangle ABC with a semicircle on each of its three sides.



If $AB = 20$, $AC = 21$, and $BC = 29$, what is the area of the shaded region?

4. If $x > 10$, what is the greatest possible value of the expression

$$(\log x)^{\log \log \log x} - (\log \log x)^{\log \log x}?$$

All the logarithms are base 10.

5. Let $\triangle ABC$ be a triangle with $AB = 3$, $BC = 4$, and $AC = 5$. Let I be the center of the circle inscribed in $\triangle ABC$. What is the product of AI, BI, and CI?

6. Two circles each have radius 1. No point is inside both circles. The circles are contained in a square. What is the area of the smallest such square? Express your answer in the form $a + b\sqrt{c}$, where a and b are positive integers and c is a square-free positive integer.

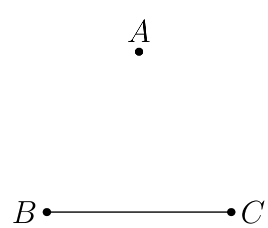
7. If z is a complex number such that

$$z + z^{-1} = \sqrt{3},$$

what is the value of

$$z^{2010} + z^{-2010}?$$

8. In the figure below, points A, B, and C are distance 6 from each other. Say that a point X is *reachable* if there is a path (not necessarily straight) connecting A and X of length at most 8 that does not intersect the interior of $\overline{BC}$. (Both X and the path must lie on the plane containing A, B, and C.) Let R be the set of reachable points. What is the area of R? Express your answer in the form $m\pi + n\sqrt{3}$, where m and n are integers.



9. Let ABC be a triangle. Let D be the midpoint of $\overline{BC}$, let E be the midpoint of $\overline{AD}$, and let F be the midpoint of $\overline{BE}$. Let G be the point where the lines AB and CF intersect. What is the value of $\dfrac{AG}{AB}$? Express your answer as a fraction in simplest form.

10. There are real numbers a and b such that for every positive number x, we have the identity

$$\tan^{-1}\left(\frac{1}{x} - \frac{x}{8}\right) + \tan^{-1}(ax) + \tan^{-1}(bx) = \frac{\pi}{2}\,.$$

(Throughout this equation, $\tan^{-1}$ means the inverse tangent function, sometimes written arctan.) What is the value of $a^2 + b^2$? Express your answer as a fraction in simplest form.

11. The sequence a_0, a_1, a_2, $\ldots$ satisfies the recurrence equation

$$a_n = 2a_{n-1} - 2a_{n-2} + a_{n-3}$$

for every integer $n \geq 3$. If $a_{20} = 1$, $a_{25} = 10$, and $a_{30} = 100$, what is the value of a_{1331}?

12. If x is a real number, let $\lfloor x \rfloor$ be the greatest integer that is less than or equal to x. If n is a positive integer, let $S(n)$ be defined by

$$S(n) = \left\lfloor \frac{n}{10^{\lfloor \log n \rfloor}} \right\rfloor + 10\left(n - 10^{\lfloor \log n \rfloor} \cdot \left\lfloor \frac{n}{10^{\lfloor \log n \rfloor}} \right\rfloor\right)\,.$$

(All the logarithms are base 10.) How many integers n from 1 to 2011 (inclusive) satisfy $S(S(n)) = n$?

13. The number 104,060,465 is divisible by a five-digit prime number. What is that prime number?

14. If $0 \le p \le 1$ and $0 \le q \le 1$, define $F(p,q)$ by

$$F(p,q) = -2pq + 3p(1-q) + 3(1-p)q - 4(1-p)(1-q).$$

Define $G(p)$ to be the maximum of $F(p,q)$ over all q (in the interval $0 \le q \le 1$). What is the value of p (in the interval $0 \le p \le 1$) that minimizes $G(p)$? Express your answer as a fraction in simplest form.

15. The game of backgammon has a "doubling" cube, which is like a standard 6-faced die except that its faces are inscribed with the numbers 2, 4, 8, 16, 32, and 64, respectively. After rolling the doubling cube four times at random, we let a be the value of the first roll, b be the value of the second roll, c be the value of the third roll, and d be the value of the fourth roll. What is the probability that $\frac{a+b}{c+d}$ is the average of $\frac{a}{c}$ and $\frac{b}{d}$? Express your answer as a fraction in simplest form.

16. Let N be the number of ordered pairs of integers (x,y) such that

$$4x^2 + 9y^2 \le 1000000000.$$

Let a be the first digit of N (from the left) and let b be the second digit of N. What is the value of $10a + b$?

17. There is a polynomial P such that for every real number x,

$$x^{512} + x^{256} + 1 = (x^2 + x + 1)P(x).$$

When P is written in standard polynomial form, how many of its coefficients are nonzero?

18. The polynomial P is a quadratic with integer coefficients. For every positive integer n, the integers $P(n)$ and $P(P(n))$ are relatively prime to n. If $P(3) = 89$, what is the value of $P(10)$?

19. If $-1 < x < 1$ and $-1 < y < 1$, define the "relativistic sum" $x \oplus y$ to be

$$x \oplus y = \frac{x + y}{1 + xy}.$$

The operation $\oplus$ is commutative and associative. Let v be the number

$$v = \frac{\sqrt[7]{17} - 1}{\sqrt[7]{17} + 1}.$$

What is the value of

$$v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v\,?$$

(In this expression, $\oplus$ appears 13 times.) Express your answer as a fraction in simplest form.

20. Let ABC be an equilateral triangle with each side of length 1. Let X be a point chosen uniformly at random on side $\overline{AB}$. Let Y be a point chosen uniformly at random on side $\overline{AC}$. (Points X and Y are chosen independently.) Let p be the probability that the distance XY is at most $\dfrac{1}{\sqrt[4]{3}}$. What is the value of $900p$, rounded to the nearest integer?