



## MATH PRIZE *for* GIRLS *at* MIT

---

### 2011 SOLUTIONS

**Problem 1** If  $m$  and  $n$  are integers such that  $3m + 4n = 100$ , what is the smallest possible value of  $|m - n|$ ?

**Answer:** 3.

**Solution:** One pair  $(m, n)$  that satisfies the constraints is  $(0, 25)$ . We get all the other pairs by repeatedly adding 4 to  $m$  and subtracting 3 from  $n$  (or subtracting 4 from  $m$  and adding 3 to  $n$ ). So the list of pairs is

$$\dots, (0, 25), (4, 22), (8, 19), (12, 16), (16, 13), (20, 10), \dots$$

The corresponding difference  $m - n$  is

$$\dots, -25, -18, -11, -4, 3, 10, \dots$$

As we see, the differences go up by 7. The smallest absolute difference is  $\boxed{3}$ .

**Problem 2** Express  $\sqrt{2 + \sqrt{3}}$  in the form  $\frac{a + \sqrt{b}}{\sqrt{c}}$ , where  $a$  is a positive integer and  $b$  and  $c$  are square-free positive integers.

**Answer:**  $\frac{1 + \sqrt{3}}{\sqrt{2}}$  or  $\frac{3 + \sqrt{3}}{\sqrt{6}}$ .

**Solution:** If  $\frac{a + \sqrt{b}}{\sqrt{c}}$  equals  $\sqrt{2 + \sqrt{3}}$ , then squaring gives

$$\frac{a^2 + b + 2a\sqrt{b}}{c} = 2 + \sqrt{3}.$$

Multiplying by  $c$  gives

$$a^2 + b + 2a\sqrt{b} = 2c + c\sqrt{3}.$$

Rearranging gives

$$a^2 - 2c + b = c\sqrt{3} - 2a\sqrt{b}.$$

To make the square of the right side an integer,  $3b$  must be a perfect square. Because  $b$  is square-free, we must have  $b = 3$ . Substituting  $b = 3$  gives

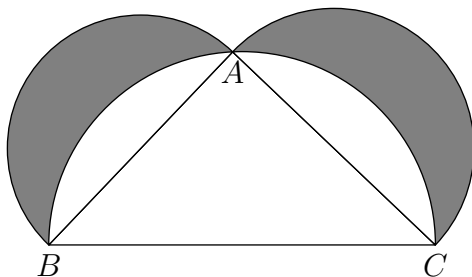
$$a^2 - 2c + 3 = (c - 2a)\sqrt{3}.$$

Because  $\sqrt{3}$  is irrational, both sides must be zero. So we have  $c = 2a$  and  $a^2 - 2c + 3 = 0$ . Substituting  $c = 2a$  into  $a^2 - 2c + 3 = 0$  gives

$$a^2 - 4a + 3 = 0.$$

So  $a$  is either 1 or 3. The corresponding value of  $c$  is 2 or 6, respectively. All the previous steps are reversible, so both solutions are valid.

**Problem 3** The figure below shows a triangle  $ABC$  with a semicircle on each of its three sides.



If  $AB = 20$ ,  $AC = 21$ , and  $BC = 29$ , what is the area of the shaded region?

**Answer:** 210.

**Solution:** On one hand, the whole region can be viewed as the (disjoint) union of the semicircle with diameter  $\overline{BC}$  and the shaded region. On the other hand, the whole region can be viewed as the (disjoint) union of the semicircle with diameter  $\overline{AB}$ , the semicircle with diameter  $\overline{AC}$ , and the triangle  $ABC$ . We know that  $\angle BAC$  is a right angle, since it is inscribed in a semicircle (or since 20-21-29 is a Pythagorean triple). By the Pythagorean theorem, the area of the semicircle with diameter  $\overline{BC}$  must equal the sum of the areas of the two semicircles with diameters  $\overline{AB}$  and  $\overline{AC}$ . The areas of the semicircles cancel out, and hence the area of the shaded region is equal to the area of triangle  $ABC$ . The area of the triangle is

$$\frac{20 \cdot 21}{2} = 10 \cdot 21 = \boxed{210}.$$

# AT MATH PRIZE FOR GIRLS 2011 SOLUTIONS

---

Note: This problem is basically the ancient problem of finding the area of the lune of Hippocrates.

**Problem 4** If  $x > 10$ , what is the greatest possible value of the expression

$$(\log x)^{\log \log \log x} - (\log \log x)^{\log \log x}?$$

All the logarithms are base 10.

**Answer:** 0.

**Solution:** The first term of the given expression is

$$\begin{aligned}(\log x)^{\log \log \log x} &= (10^{\log \log x})^{\log \log \log x} \\ &= 10^{(\log \log x)(\log \log \log x)}.\end{aligned}$$

The second term is

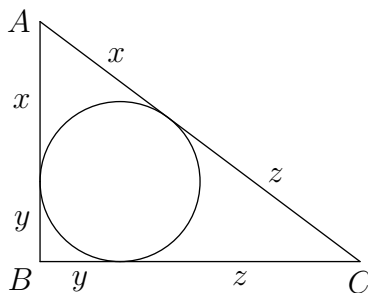
$$\begin{aligned}(\log \log x)^{\log \log x} &= (10^{\log \log \log x})^{\log \log x} \\ &= 10^{(\log \log \log x)(\log \log x)} \\ &= 10^{(\log \log x)(\log \log \log x)}.\end{aligned}$$

The two terms are equal! So their difference is always  $\boxed{0}$ .

**Problem 5** Let  $\triangle ABC$  be a triangle with  $AB = 3$ ,  $BC = 4$ , and  $AC = 5$ . Let  $I$  be the center of the circle inscribed in  $\triangle ABC$ . What is the product of  $AI$ ,  $BI$ , and  $CI$ ?

**Answer:** 10.

**Solution:** Below is a picture of  $\triangle ABC$  and its inscribed circle.

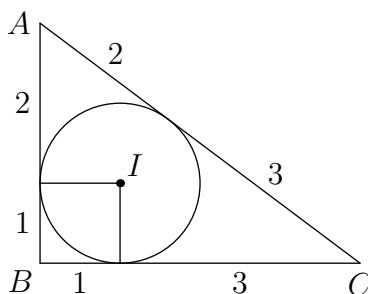


# AT MATH PRIZE FOR GIRLS 2011      SOLUTIONS

---

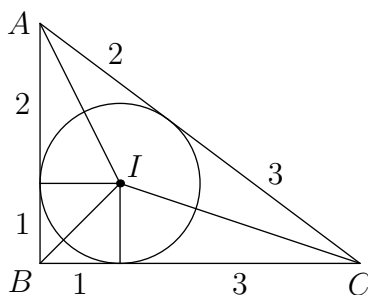
Because the tangents from  $A$  to the circle have the same length, we called both lengths  $x$ . We did something similar for the other tangents. Because  $AB = 3$ , we have  $x + y = 3$ . Similarly  $y + z = 4$  and  $x + z = 5$ . Solving these three equations gives  $x = 2$ ,  $y = 1$ , and  $z = 3$ .

Let's substitute these numbers into our picture. At the same time, let's draw a couple of radii.



Because  $\triangle ABC$  is a 3-4-5 triangle,  $\angle B$  is a right angle. The two displayed radii form right angles with their tangents. So the displayed quadrilateral is actually a rectangle. In fact, since two of its sides have length 1, the quadrilateral is a square of side length 1. In particular, the radius of the circle is 1.

Next, let's draw segments from the incenter  $I$  to the vertices.



By the Pythagorean theorem, we have  $AI = \sqrt{2^2 + 1^2} = \sqrt{5}$ . Similarly, we have  $BI = \sqrt{1^2 + 1^2} = \sqrt{2}$ . Similarly, we have  $CI = \sqrt{3^2 + 1^2} = \sqrt{10}$ . The product of these three lengths is

$$\sqrt{5} \cdot \sqrt{2} \cdot \sqrt{10} = \sqrt{10} \cdot \sqrt{10} = \boxed{10}.$$

**Problem 6** Two circles each have radius 1. No point is inside both circles. The circles are contained in a square. What is the area of the smallest such

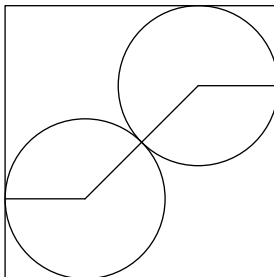
# AT MATH PRIZE FOR GIRLS 2011      SOLUTIONS

---

square? Express your answer in the form  $a + b\sqrt{c}$ , where  $a$  and  $b$  are positive integers and  $c$  is a square-free positive integer.

**Answer:**  $6 + 4\sqrt{2}$ .

**Solution:** Consider the picture below of two circles in a square.



Each circle has radius 1. The segment joining the two centers forms a  $45^\circ$  angle with the horizontal and vertical lines. The distance between the two centers is 2. So the horizontal distance between the two centers is  $\sqrt{2}$ . Hence the side length of the square is  $2 + \sqrt{2}$ . Thus the area of the square is

$$(2 + \sqrt{2})^2 = 4 + 4\sqrt{2} + 2 = 6 + 4\sqrt{2}.$$

We will now show that the configuration above is optimal. Consider any configuration of two circles in a square satisfying the conditions of the problem. Let  $s$  be the side length of the square. We may impose a coordinate system on the configuration, so that the vertices of the square are  $(0, 0)$ ,  $(s, 0)$ ,  $(s, s)$ , and  $(0, s)$ . Because the two circles of radius 1 are contained in the square, the centers of the circles have  $x$ -coordinates and  $y$ -coordinates between 1 and  $s - 1$ . So the distance between the two centers is at most  $(s - 2)\sqrt{2}$ . Because the two circles are disjoint, the distance between the two centers is at least 2. So we get the inequality  $(s - 2)\sqrt{2} \geq 2$ . Solving for  $s$  gives  $s \geq 2 + \sqrt{2}$ . So the area of the square is at least  $(2 + \sqrt{2})^2$ , which is  $6 + 4\sqrt{2}$ .

**Problem 7** If  $z$  is a complex number such that

$$z + z^{-1} = \sqrt{3},$$

what is the value of

$$z^{2010} + z^{-2010}?$$

**Answer:**  $-2$ .

# AT MATH PRIZE FOR GIRLS 2011      SOLUTIONS

---

**Solution:** We are given that

$$z + z^{-1} = \sqrt{3} = 2 \cdot \frac{\sqrt{3}}{2} = 2 \cos \frac{\pi}{6}.$$

Hence  $z$  is either  $\operatorname{cis} \frac{\pi}{6}$  or  $\operatorname{cis}(-\frac{\pi}{6})$ .

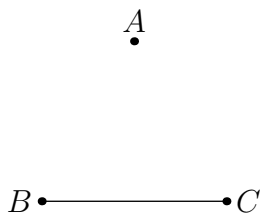
Let's assume that  $z$  is  $\operatorname{cis} \frac{\pi}{6}$ . (The other case is similar.) We have

$$z^{2010} = \left(\operatorname{cis} \frac{\pi}{6}\right)^{2010} = \operatorname{cis} \frac{2010\pi}{6} = \operatorname{cis}(335\pi) = \operatorname{cis} \pi = -1.$$

So  $z^{-2010}$  is  $-1$  also. Thus

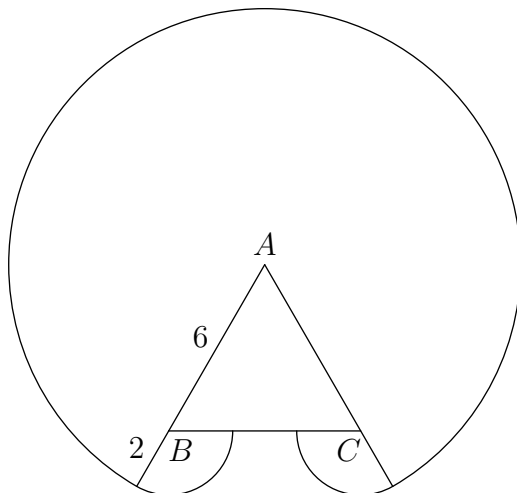
$$z^{2010} + z^{-2010} = -1 + (-1) = \boxed{-2}.$$

**Problem 8** In the figure below, points  $A$ ,  $B$ , and  $C$  are distance 6 from each other. Say that a point  $X$  is *reachable* if there is a path (not necessarily straight) connecting  $A$  and  $X$  of length at most 8 that does not intersect the interior of  $\overline{BC}$ . (Both  $X$  and the path must lie on the plane containing  $A$ ,  $B$ , and  $C$ .) Let  $R$  be the set of reachable points. What is the area of  $R$ ? Express your answer in the form  $m\pi + n\sqrt{3}$ , where  $m$  and  $n$  are integers.



**Answer:**  $56\pi + 9\sqrt{3}$ .

**Solution:** The picture below shows all the reachable points.



Each point in  $\triangle ABC$  is reachable, since the straight path from that point to  $A$  is of length at most 6. Each point in the displayed sector with center  $A$  and radius 8 is reachable, since the straight path from that point to  $A$  is of length at most 8. Each point in the displayed sector with center  $B$  and radius 2 is reachable, since the path from that point to  $B$  and then to  $A$  is of length at most  $2 + 6 = 8$ . Similarly, each point in the displayed sector with center  $C$  and radius 2 is reachable. No point outside of these regions is reachable.

The sector with center  $A$  and radius 8 is a  $300^\circ$  sector. Hence the sector is  $\frac{5}{6}$  of a circle. Its area is thus  $\frac{5}{6}\pi(8)^2$ , which simplifies to  $\frac{160}{3}\pi$ . The sector with center  $B$  and radius 2 is a  $120^\circ$  sector. Hence the sector is  $\frac{1}{3}$  of a circle. Its area is thus  $\frac{1}{3}\pi(2)^2$ , which simplifies to  $\frac{4}{3}\pi$ . Similarly, the sector with center  $C$  has area  $\frac{4}{3}\pi$ . The total area of the three sectors is therefore

$$\frac{160\pi}{3} + \frac{4\pi}{3} + \frac{4\pi}{3} = \frac{168\pi}{3} = 56\pi.$$

By the “2-3-4” formula, the area of the equilateral triangle  $ABC$  (with side length 6) is

$$\frac{(6)^2\sqrt{3}}{4} = \frac{36\sqrt{3}}{4} = 9\sqrt{3}.$$

So the area of the reachable region is  $\boxed{56\pi + 9\sqrt{3}}$ .

**Problem 9** Let  $ABC$  be a triangle. Let  $D$  be the midpoint of  $\overline{BC}$ , let  $E$  be the midpoint of  $\overline{AD}$ , and let  $F$  be the midpoint of  $\overline{BE}$ . Let  $G$  be the point

# AT MATH PRIZE FOR GIRLS 2011      SOLUTIONS

---

where the lines  $AB$  and  $CF$  intersect. What is the value of  $\frac{AG}{AB}$ ? Express your answer as a fraction in simplest form.

**Answer:**  $\frac{5}{7}$ .

**Solution:** We will use mass points to solve this problem. Because  $D$  is the midpoint of  $\overline{BC}$ , we have

$$D = \frac{1}{2}B + \frac{1}{2}C.$$

Because  $E$  is the midpoint of  $\overline{AD}$ , we have

$$E = \frac{1}{2}A + \frac{1}{2}D = \frac{1}{2}A + \frac{1}{2}\left(\frac{1}{2}B + \frac{1}{2}C\right) = \frac{1}{2}A + \frac{1}{4}B + \frac{1}{4}C.$$

Because  $F$  is the midpoint of  $\overline{BE}$ , we have

$$F = \frac{1}{2}B + \frac{1}{2}E = \frac{1}{2}B + \frac{1}{2}\left(\frac{1}{2}A + \frac{1}{4}B + \frac{1}{4}C\right) = \frac{1}{4}A + \frac{5}{8}B + \frac{1}{8}C.$$

We can rewrite  $F$  as follows:

$$F = \frac{7}{8}\left(\frac{2}{7}A + \frac{5}{7}B\right) + \frac{1}{8}C.$$

Define  $G'$  to be  $\frac{2}{7}A + \frac{5}{7}B$ . By definition,  $G'$  is on  $\overline{AB}$ . On the other hand, from the displayed equation above, we have

$$F = \frac{7}{8}G' + \frac{1}{8}C.$$

So  $G'$  is on the line  $CF$ . Hence  $G'$  must be the point  $G$  where lines  $AB$  and  $CF$  intersect. Looking at the definition of  $G'$ , we have  $AG = \frac{5}{7}AB$ . Hence  $\frac{AG}{AB}$  is  $\frac{5}{7}$ .

**Problem 10** There are real numbers  $a$  and  $b$  such that for every positive number  $x$ , we have the identity

$$\tan^{-1}\left(\frac{1}{x} - \frac{x}{8}\right) + \tan^{-1}(ax) + \tan^{-1}(bx) = \frac{\pi}{2}.$$

(Throughout this equation,  $\tan^{-1}$  means the inverse tangent function, sometimes written  $\arctan$ .) What is the value of  $a^2 + b^2$ ? Express your answer as a fraction in simplest form.

**Answer:**  $\frac{3}{4}$ .



# AT MATH PRIZE FOR GIRLS 2011 SOLUTIONS

---

**Solution:** Rearranging the given equation, we have

$$\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{x} - \frac{x}{8}\right) = \tan^{-1}(ax) + \tan^{-1}(bx).$$

The tangent of the left side is

$$\tan\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{x} - \frac{x}{8}\right)\right) = \frac{1}{\tan(\tan^{-1}(\frac{1}{x} - \frac{x}{8}))} = \frac{1}{\frac{1}{x} - \frac{x}{8}} = \frac{x}{1 - \frac{1}{8}x^2}.$$

By the tangent addition formula, the tangent of the right side is

$$\tan(\tan^{-1}(ax) + \tan^{-1}(bx)) = \frac{ax + bx}{1 - (ax)(bx)} = \frac{(a + b)x}{1 - abx^2}.$$

In order for both sides to be equal for every positive number  $x$ , we must have  $a + b = 1$  and  $ab = \frac{1}{8}$ . Hence

$$a^2 + b^2 = (a + b)^2 - 2ab = 1^2 - 2\left(\frac{1}{8}\right) = 1 - \frac{1}{4} = \boxed{\frac{3}{4}}.$$

**Problem 11** The sequence  $a_0, a_1, a_2, \dots$  satisfies the recurrence equation

$$a_n = 2a_{n-1} - 2a_{n-2} + a_{n-3}$$

for every integer  $n \geq 3$ . If  $a_{20} = 1$ ,  $a_{25} = 10$ , and  $a_{30} = 100$ , what is the value of  $a_{1331}$ ?

**Answer:** 181.

**Solution:** Let's calculate the first few values of the sequence, looking for a pattern. Plugging  $n = 3$  into the recurrence equation, we get

$$a_3 = 2a_2 - 2a_1 + a_0.$$

Plugging in  $n = 4$  gives

$$a_4 = 2a_3 - 2a_2 + a_1 = 2(2a_2 - 2a_1 + a_0) - 2a_2 + a_1 = 2a_2 - 3a_1 + 2a_0.$$

Plugging in  $n = 5$  gives

$$\begin{aligned} a_5 &= 2a_4 - 2a_3 + a_2 \\ &= 2(2a_2 - 3a_1 + 2a_0) - 2(2a_2 - 2a_1 + a_0) + a_2 \\ &= a_2 - 2a_1 + 2a_0. \end{aligned}$$

# AT MATH PRIZE FOR GIRLS 2011 SOLUTIONS

---

Plugging in  $n = 6$  gives

$$\begin{aligned} a_6 &= 2a_5 - 2a_4 + a_3 \\ &= 2(a_2 - 2a_1 + 2a_0) - 2(2a_2 - 3a_1 + 2a_0) + (2a_2 - 2a_1 + a_0) \\ &= a_0. \end{aligned}$$

Aha! We discovered that  $a_6 = a_0$ . A similar calculation reveals that  $a_7 = a_1$ ,  $a_8 = a_2$ , and so on. The sequence is 6-periodic.

Because the sequence is 6-periodic, we have  $a_2 = a_{20} = 1$ . Similarly, we have  $a_1 = a_{25} = 10$ . Also, we have  $a_0 = 100$ . Finally, we have  $a_{1331} = a_5$ . According to our work above, we have

$$a_{1331} = a_5 = a_2 - 2a_1 + 2a_0 = 1 - 2(10) + 2(100) = 1 - 20 + 200 = \boxed{181}.$$

**Problem 12** If  $x$  is a real number, let  $\lfloor x \rfloor$  be the greatest integer that is less than or equal to  $x$ . If  $n$  is a positive integer, let  $S(n)$  be defined by

$$S(n) = \left\lfloor \frac{n}{10^{\lfloor \log n \rfloor}} \right\rfloor + 10 \left( n - 10^{\lfloor \log n \rfloor} \cdot \left\lfloor \frac{n}{10^{\lfloor \log n \rfloor}} \right\rfloor \right).$$

(All the logarithms are base 10.) How many integers  $n$  from 1 to 2011 (inclusive) satisfy  $S(S(n)) = n$ ?

**Answer:** 108.

**Solution:** The expression  $\lfloor \log n \rfloor$  is 1 less than the number of digits of  $n$ . So  $\left\lfloor \frac{n}{10^{\lfloor \log n \rfloor}} \right\rfloor$  is the first digit (from the left) of  $n$ . Hence

$$n - 10^{\lfloor \log n \rfloor} \cdot \left\lfloor \frac{n}{10^{\lfloor \log n \rfloor}} \right\rfloor$$

is just  $n$  with the first digit of  $n$  erased. Therefore,  $S(n)$  itself is a cyclic shift (1 digit to the left) of the digits of  $n$ . For example,  $S(1234) = 2341$ . Also,  $S(1034) = 0341 = 341$ .

If the second digit of  $n$  is nonzero (or if  $n$  has just one digit), then  $S(S(n))$  is a cyclic shift (2 digits to the left) of the digits of  $n$ . For example,  $S(S(1234)) = 3412$ . But if the second digit of  $n$  is zero, then  $S(n)$  and hence  $S(S(n))$  have fewer digits than  $n$ . For example,  $S(S(1034)) = 413$ .

Now let's count  $n$  (with  $1 \leq n \leq 2011$ ) such that  $S(S(n)) = n$ . We divide into cases depending on how many digits  $n$  has.

# AT MATH PRIZE FOR GIRLS 2011      SOLUTIONS

---

**Case  $n$  is a 1-digit integer (with  $n \geq 1$ ):** Then  $S(n) = n$ , so  $S(S(n)) = n$ . In other words, all positive 1-digit integers contribute, giving a count of 9.

**Case  $n$  is a 2-digit integer  $ab$ :** We have  $S(ab) = ba$ . If  $b$  is nonzero, then  $S(S(ab)) = ab$ . But if  $b$  is zero, then  $S(S(ab))$  will have fewer than 2 digits. So we're counting the two-digit integers whose digits are both nonzero. The count is  $9 \cdot 9$ , or 81.

**Case  $n$  is a 3-digit integer  $abc$ :** We have  $S(abc) = bca$ . Again if  $b$  were zero, then  $S(S(abc))$  will have too few digits. So  $b$  is nonzero. Then  $S(S(abc)) = S(bca) = cab$ . We need  $abc = cab$ , which means  $a = b = c$ . The total count is 9.

**Case  $n$  is a 4-digit integer  $abcd$  (with  $n \leq 2011$ ):** We have  $S(abcd) = bcda$ . Again, we must have  $b$  nonzero. Then  $S(S(abcd)) = S(bcda) = cdab$ . So we must have  $abcd = cdab$ . Comparing digit by digit, we must have  $a = c$  and  $b = d$ . Because  $n$  is at most 2011, we must have  $a = c = 1$ . We have 9 choices for  $b$ . The total count is 9.

Adding all four cases, we get a final count of  $9 + 81 + 9 + 9$ , which is 108.

**Problem 13** The number 104,060,465 is divisible by a five-digit prime number. What is that prime number?

**Answer:** 10,613.

**Solution:** The number 104,060,465 looks like Row 4 of Pascal's triangle: 1, 4, 6, 4, 1. In particular, we have

$$104,060,465 = 104,060,401 + 64 = 101^4 + 64 = (101^2)^2 + 64 = 10201^2 + 8^2.$$

Let's try to complete the square:

$$\begin{aligned} 104,060,465 &= 10201^2 + 8^2 \\ &= (10201^2 + 2(10201)8 + 8^2)^2 - 2(10201)8 \\ &= (10201 + 8)^2 - 16(10201) \\ &= 10209^2 - 4^2 \cdot 101^2 \\ &= 10209^2 - 404^2. \end{aligned}$$

We can now apply difference of squares:

$$\begin{aligned} 104,060,465 &= 10209^2 - 404^2 \\ &= (10209 - 404)(10209 + 404) \\ &= (9805)(10613). \end{aligned}$$

# AT MATH PRIZE FOR GIRLS 2011 SOLUTIONS

---

The five-digit prime divisor of 104,060,465 can't divide 9805, and so must divide 10613. But the only five-digit divisor of 10613 is itself, so the five-digit prime divisor is  $\boxed{10613}$ .

Here is an alternative solution. We will start with our first observation above:

$$104,060,465 = 101^4 + 64.$$

We now recall the identity due to Sophie Germain:

$$a^4 + 4b^4 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2).$$

(One way to prove Germain's identity is to add and subtract  $4a^2b^2$  on the left side.) Setting  $a = 101$  and  $b = 2$ , we get

$$\begin{aligned} 101^4 + 64 &= [101^2 + 2(101)2 + 2(2)^2] \cdot [101^2 - 2(101)2 + 2(2)^2] \\ &= (10201 + 404 + 8)(10201 - 404 + 8) \\ &= (9805)(10613). \end{aligned}$$

As before, the five-digit prime number must be  $\boxed{10613}$ .

**Problem 14** If  $0 \leq p \leq 1$  and  $0 \leq q \leq 1$ , define  $F(p, q)$  by

$$F(p, q) = -2pq + 3p(1 - q) + 3(1 - p)q - 4(1 - p)(1 - q).$$

Define  $G(p)$  to be the maximum of  $F(p, q)$  over all  $q$  (in the interval  $0 \leq q \leq 1$ ). What is the value of  $p$  (in the interval  $0 \leq p \leq 1$ ) that minimizes  $G(p)$ ? Express your answer as a fraction in simplest form.

**Answer:**  $\frac{7}{12}$ .

**Solution:** For fixed  $p$ , the expression for  $F(p, q)$  is linear in  $q$ . Hence  $G(p)$  is the maximum of the end values  $F(p, 0)$  and  $F(p, 1)$ . Plugging in  $q = 0$ , we see that  $F(p, 0)$  is

$$F(p, 0) = 3p - 4(1 - p) = 7p - 4.$$

Plugging in  $q = 1$ , we see that  $F(p, 1)$  is

$$F(p, 1) = -2p + 3(1 - p) = 3 - 5p.$$

Hence  $G(p)$  is  $\max(7p - 4, 3 - 5p)$ .

# AT MATH PRIZE FOR GIRLS 2011      SOLUTIONS

---

To get a feel for  $G(p)$ , let's compare the two expressions  $7p - 4$  and  $3 - 5p$ . When  $p = \frac{7}{12}$ , the two expressions are equal. When  $p < \frac{7}{12}$ , the second expression is greater. When  $p > \frac{7}{12}$ , the first expression is greater. So  $G$  is linear on  $[0, \frac{7}{12}]$  and is linear on  $[\frac{7}{12}, 1]$ . We calculate that  $G(0) = 3$ ,  $G(\frac{7}{12}) = \frac{1}{12}$ , and  $G(1) = 3$ . So the minimum value of  $G(p)$  occurs when  $p$  is  $\boxed{\frac{7}{12}}$ .

**Problem 15** The game of backgammon has a “doubling” cube, which is like a standard 6-faced die except that its faces are inscribed with the numbers 2, 4, 8, 16, 32, and 64, respectively. After rolling the doubling cube four times at random, we let  $a$  be the value of the first roll,  $b$  be the value of the second roll,  $c$  be the value of the third roll, and  $d$  be the value of the fourth roll. What is the probability that  $\frac{a+b}{c+d}$  is the average of  $\frac{a}{c}$  and  $\frac{b}{d}$ ? Express your answer as a fraction in simplest form.

**Answer:**  $\frac{163}{648}$ .

**Solution:** The average of  $\frac{a}{c}$  and  $\frac{b}{d}$  is

$$\frac{1}{2} \left( \frac{a}{c} + \frac{b}{d} \right) = \frac{ad + bc}{2cd}.$$

So for  $\frac{a+b}{c+d}$  to equal this average means

$$\frac{a+b}{c+d} = \frac{ad + bc}{2cd}.$$

Clearing fractions gives

$$2(a+b)cd = (ad + bc)(c+d) = acd + ad^2 + bc^2 + bcd.$$

Putting all the terms on one side gives

$$ad^2 - acd - bcd + bc^2 = 0.$$

Factoring gives

$$(ad - bc)(d - c) = 0.$$

All the steps above are reversible. So  $\frac{a+b}{c+d}$  is the average of  $\frac{a}{c}$  and  $\frac{b}{d}$  if and only if  $ad = bc$  or  $c = d$ .

# AT MATH PRIZE FOR GIRLS 2011 SOLUTIONS

---

Let's compute separately the probability that  $ad = bc$  and the probability that  $c = d$ . First, the probability that  $c = d$  is  $\frac{1}{6}$ .

Let's now compute the probability that  $ad = bc$ . The expression  $ad$  has the possible values

$$2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}, 2^{11}, 2^{12}$$

with respective probabilities

$$\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}.$$

The same is true of the expression  $bc$ . So the probability that  $ad = bc$  is

$$\begin{aligned} \Pr(ad = bc) &= \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2}{36^2} \\ &= \frac{2(1^2 + 2^2 + 3^2 + 4^2 + 5^2) + 6^2}{1296} \\ &= \frac{2(1 + 4 + 9 + 16 + 25) + 36}{1296} \\ &= \frac{2(55) + 36}{1296} \\ &= \frac{146}{1296} \\ &= \frac{73}{648}. \end{aligned}$$

To avoid double-counting, we also have to compute the probability that  $ad = bc$  and  $c = d$ . The only way for that to happen is if  $a = b$  and  $c = d$ . So the probability is  $\frac{1}{6} \cdot \frac{1}{6}$ , which is  $\frac{1}{36}$ .

By the Principle of Inclusion-Exclusion, the probability that  $ad = bc$  or  $c = d$  is

$$\frac{73}{648} + \frac{1}{6} - \frac{1}{36} = \frac{73}{648} + \frac{108}{648} - \frac{18}{648} = \frac{73 + 108 - 18}{648} = \boxed{\frac{163}{648}}.$$

**Problem 16** Let  $N$  be the number of ordered pairs of integers  $(x, y)$  such that

$$4x^2 + 9y^2 \leq 1000000000.$$

# AT MATH PRIZE FOR GIRLS 2011 SOLUTIONS

---

Let  $a$  be the first digit of  $N$  (from the left) and let  $b$  be the second digit of  $N$ . What is the value of  $10a + b$ ?

**Answer:** 52.

**Solution:** The value of  $N$  is the number of lattice points (points with integer coordinates) inside the ellipse  $(2x)^2 + (3y)^2 \leq 1,000,000,000$ . Imagine a unit square at the center of each such lattice point. Then  $N$  is the number of such unit squares. Equivalently,  $N$  is the area of the union of all such unit squares. The union of such unit squares is close in area to the ellipse. The area of the ellipse is  $\frac{1,000,000,000}{6}\pi$ , which is approximately

$$\frac{1,000,000,000}{6}(3.1416) = \frac{3,141,600,000}{6} = 523,600,000.$$

So  $N$  is approximately 523,600,000. Assuming the approximation is close enough, that means  $a$  is 5 and  $b$  is 2. So  $10a + b$  is 52.

In the rest of the solution, we will prove that the approximation is close enough. Imagine stretching the entire picture by a factor of 2 in the  $x$  direction and by a factor of 3 in the  $y$  direction. The ellipse transforms into a circle with radius  $\sqrt{10^9}$ . Each unit square transforms into a 2-by-3 rectangle. The union of these  $N$  rectangles has area  $6N$ . In each such rectangle, the distance of every point in the rectangle to the center of the rectangle is at most

$$\sqrt{1^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{13}{4}}.$$

So the union of the rectangles is contained in a big circle of radius  $\sqrt{10^9} + \sqrt{\frac{13}{4}}$ . Similarly, the union of the rectangles contains a small circle of radius  $\sqrt{10^9} - \sqrt{\frac{13}{4}}$ .

The area of the big circle is

$$\pi \left( \sqrt{10^9} + \sqrt{\frac{13}{4}} \right)^2 = \pi \left( 10^9 + 10^4\sqrt{130} + \frac{13}{4} \right) < 10^9\pi + 4 \cdot 10^4 \cdot 12 + 13 < 3,150,000.$$

The area of the small circle is

$$\pi \left( \sqrt{10^9} - \sqrt{\frac{13}{4}} \right)^2 = \pi \left( 10^9 - 10^4\sqrt{130} + \frac{13}{4} \right) > 10^9\pi - 4 \cdot 10^4 \cdot 12 > 3,120,000.$$

# AT MATH PRIZE FOR GIRLS 2011 SOLUTIONS

---

That means  $6N$ , the area of the union of the rectangles, is between 3,120,000 and 3,150,000. So  $N$  itself is between 520,000 and 525,000. Thus our approximation is close enough.

**Problem 17** There is a polynomial  $P$  such that for every real number  $x$ ,

$$x^{512} + x^{256} + 1 = (x^2 + x + 1)P(x).$$

When  $P$  is written in standard polynomial form, how many of its coefficients are nonzero?

**Answer:** 341.

**Solution:** We can cleverly rewrite  $x^{512} + x^{256} + 1$  as follows:

$$\begin{aligned} x^{512} + x^{256} + 1 &= (x^{512} + x^{511} + \cdots + 1) \\ &\quad - (x^{511} + x^{510} + \cdots + x^{257}) \\ &\quad - (x^{255} + x^{254} + \cdots + x). \end{aligned}$$

Each of the three parenthesized polynomials is a multiple of  $x^2 + x + 1$ . For example,

$$x^{512} + x^{511} + \cdots + 1 = (x^2 + x + 1)(x^{510} + x^{507} + \cdots + 1).$$

(The exponents in the rightmost factor skip down by 3. So that factor has 171 terms.) Similarly, we have

$$x^{511} + x^{510} + \cdots + x^{257} = (x^2 + x + 1)(x^{509} + x^{506} + \cdots + x^{257}).$$

(The rightmost factor has 85 terms.) Finally, we have

$$x^{255} + x^{254} + \cdots + x = (x^2 + x + 1)(x^{253} + x^{250} + \cdots + x).$$

(The rightmost factor has 85 terms.) By subtracting the last two equations from the equation before that, we get a factorization of  $x^{512} + x^{256} + 1$  as the product of  $x^2 + x + 1$  and a polynomial with  $171 + 85 + 85 = 341$  terms. The terms all have different exponents. So the answer to the problem is 341.

**Problem 18** The polynomial  $P$  is a quadratic with integer coefficients. For every positive integer  $n$ , the integers  $P(n)$  and  $P(P(n))$  are relatively prime to  $n$ . If  $P(3) = 89$ , what is the value of  $P(10)$ ?

**Answer:** 859.



# AT MATH PRIZE FOR GIRLS 2011 SOLUTIONS

---

**Solution:** Throughout we will use the fact that if  $a \equiv b \pmod{n}$ , then  $P(a) \equiv P(b) \pmod{n}$ .

First, we will determine  $P(0)$ . For every positive integer  $n$ , we have

$$0 \equiv n \pmod{n}.$$

Applying  $P$  on both sides, we have

$$P(0) \equiv P(n) \pmod{n}.$$

Because  $P(n)$  and  $n$  are relatively prime,  $P(0)$  and  $n$  are relatively prime. Since this is true for every positive integer  $n$  (and in particular, for every prime),  $P(0)$  is either  $-1$  or  $1$ . To determine which one, we work mod 3:

$$P(0) \equiv P(3) = 89 \equiv 2 \pmod{3}.$$

Hence  $P(0) = -1$ .

Next, we will determine  $P(-1)$ . Recall the equation

$$P(0) \equiv P(n) \pmod{n}.$$

Because  $P(0) = -1$ , we have

$$-1 \equiv P(n) \pmod{n}.$$

Applying  $P$  on both sides, we have

$$P(-1) \equiv P(P(n)) \pmod{n}.$$

Because  $P(P(n))$  and  $n$  are relatively prime,  $P(-1)$  and  $n$  are relatively prime. As before,  $P(-1)$  must be either  $-1$  or  $1$ . To determine which one, we use the fact that

$$-1 \equiv 3 \pmod{4}.$$

Applying  $P$  on both sides, we get

$$P(-1) \equiv P(3) = 89 \equiv 1 \pmod{4}.$$

Hence  $P(-1)$  is  $1$ .

We have discovered that  $P(0) = -1$ ,  $P(-1) = 1$ , and  $P(3) = 89$ . There is a unique quadratic that satisfies all three equations. One way to find this quadratic is to write  $P(x)$  as follows:

$$P(x) = a + bx + cx(x + 1).$$

## AT MATH PRIZE FOR GIRLS 2011 SOLUTIONS

---

Plugging in  $x = 0$ , we find  $a = -1$ . Plugging in  $x = -1$ , we find  $b = -2$ . Plugging in  $x = 3$ , we find  $c = 8$ . So the polynomial  $P$  is

$$P(x) = a + bx + cx(x + 1) = -1 - 2x + 8x(x + 1) = 8x^2 + 6x - 1.$$

In particular, we have

$$P(10) = 8(10)^2 + 6(10) - 1 = 800 + 60 - 1 = \boxed{859}.$$

**Problem 19** If  $-1 < x < 1$  and  $-1 < y < 1$ , define the “relativistic sum”  $x \oplus y$  to be

$$x \oplus y = \frac{x + y}{1 + xy}.$$

The operation  $\oplus$  is commutative and associative. Let  $v$  be the number

$$v = \frac{\sqrt[7]{17} - 1}{\sqrt[7]{17} + 1}.$$

What is the value of

$$v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v?$$

(In this expression,  $\oplus$  appears 13 times.) Express your answer as a fraction in simplest form.

**Answer:**  $\frac{144}{145}$ .

**Solution:** The number  $v$  is of the form  $\frac{a-1}{a+1}$ , where  $a = \sqrt[7]{17}$ . If  $b$  and  $c$  are positive numbers, then

$$\begin{aligned} \frac{b-1}{b+1} \oplus \frac{c-1}{c+1} &= \frac{\frac{b-1}{b+1} + \frac{c-1}{c+1}}{1 + \frac{b-1}{b+1} \cdot \frac{c-1}{c+1}} \\ &= \frac{(b-1)(c+1) + (b+1)(c-1)}{(b+1)(c+1) + (b-1)(c-1)} \\ &= \frac{bc - c + b - 1 + bc + c - b - 1}{bc + b + c + 1 + bc - b - c + 1} \\ &= \frac{2bc - 2}{2bc + 2} \\ &= \frac{bc - 1}{bc + 1}. \end{aligned}$$

# AT MATH PRIZE FOR GIRLS 2011 SOLUTIONS

---

In other words, if we define  $f(t) = \frac{t-1}{t+1}$  for any positive number  $t$ , then  $f(b) \oplus f(c) = f(bc)$ . We can extend this equation to more terms. For example, if  $b$ ,  $c$ , and  $d$  are positive, then

$$f(b) \oplus f(c) \oplus f(d) = f(bc) \oplus f(d) = f(bcd).$$

(This reasoning also explains why  $\oplus$  is associative.) And so on.

Going back to our original problem, we have

$$\begin{aligned} \underbrace{v \oplus v \oplus \cdots \oplus v}_{14 \text{ copies of } v} &= \underbrace{f(a) \oplus f(a) \oplus \cdots \oplus f(a)}_{14 \text{ copies of } f(a)} \\ &= f(\underbrace{a \times a \times \cdots \times a}_{14 \text{ copies of } a}) \\ &= f(a^{14}) \\ &= f(17^2) \\ &= f(289) \\ &= \frac{289 - 1}{289 + 1} \\ &= \frac{288}{290} \\ &= \frac{144}{145}. \end{aligned}$$

Note: Our “relativistic sum” definition was inspired by the addition law for velocities in special relativity.

**Problem 20** Let  $ABC$  be an equilateral triangle with each side of length 1. Let  $X$  be a point chosen uniformly at random on side  $\overline{AB}$ . Let  $Y$  be a point chosen uniformly at random on side  $\overline{AC}$ . (Points  $X$  and  $Y$  are chosen independently.) Let  $p$  be the probability that the distance  $XY$  is at most  $\frac{1}{\sqrt[4]{3}}$ . What is the value of  $900p$ , rounded to the nearest integer?

**Answer:** 628.

**Solution:** Let  $x = AX$  and  $y = AY$ . By the Law of Cosines, the distance squared between  $X$  and  $Y$  is

$$x^2 + y^2 - 2xy \cos 60^\circ = x^2 + y^2 - 2xy \cdot \frac{1}{2} = x^2 + y^2 - xy.$$

# AT MATH PRIZE FOR GIRLS 2011      SOLUTIONS

---

So  $XY \leq \frac{1}{\sqrt[4]{3}}$  is equivalent to

$$x^2 + y^2 - xy \leq \frac{1}{\sqrt{3}}.$$

The graph of this inequality is an ellipse, whose axes are rotated  $45^\circ$  from the coordinate axes. If  $XY \leq \frac{1}{\sqrt[4]{3}}$ , then  $XY < \frac{\sqrt{3}}{2}$ , so  $X$  can't be the vertex  $C$ , which means that  $x$  can't be 1. As a consequence, the ellipse doesn't intersect the line  $x = 1$ . Similarly, the ellipse doesn't intersect the line  $y = 1$ . So the portion of the ellipse in the first quadrant lies in the unit square. The area of that portion of the ellipse is the value of  $p$ .

The ellipse inequality

$$x^2 + y^2 - xy \leq \frac{1}{\sqrt{3}}$$

is equivalent to the inequality

$$\left(\frac{x+y}{\sqrt{2}}\right)^2 + 3\left(\frac{x-y}{\sqrt{2}}\right)^2 \leq \frac{2}{\sqrt{3}}.$$

Let's rotate the entire graph by  $45^\circ$  clockwise. Then the original ellipse transforms to the axes-parallel ellipse

$$x^2 + 3y^2 \leq \frac{2}{\sqrt{3}}.$$

The boundary lines  $x = 0$  and  $y = 0$  transform to the lines  $y = x$  and  $y = -x$ . Now let's transform the new ellipse to a circle by stretching the entire figure in the  $y$  direction by a factor of  $\sqrt{3}$ . The axes-parallel ellipse transforms to the circle

$$x^2 + y^2 \leq \frac{2}{\sqrt{3}}.$$

The original boundary lines  $y = x$  and  $y = -x$  transform to the lines  $y = \sqrt{3}x$  and  $y = -\sqrt{3}x$ . These new boundary lines form an angle of  $120^\circ$ . So the area of the circle sector inside the boundary lines is

$$\frac{1}{3} \cdot \pi \cdot \frac{2}{\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}.$$

# AT MATH PRIZE FOR GIRLS 2011 SOLUTIONS

---

Because of the stretch, the area of the portion of the ellipse inside the boundary lines is

$$\frac{1}{\sqrt{3}} \cdot \frac{2\pi}{3\sqrt{3}} = \frac{2\pi}{9}.$$

Thus  $p = \frac{2\pi}{9}$ . Hence  $900p$  is  $200\pi$ , which rounds to 628.