MATH PRIZE *for* GIRLS *at* MIT

2011 Solutions

Problem 1 If m and n are integers such that $3m + 4n = 100$, what is the smallest possible value of $|m - n|$?

Answer: 3.

Solution: One pair (m, n) that satisfies the constraints is $(0, 25)$. We get all the other pairs by repeatedly adding 4 to m and subtracting 3 from n (or subtracting 4 from m and adding 3 to n). So the list of pairs is

$$\ldots, (0, 25), (4, 22), (8, 19), (12, 16), (16, 13), (20, 10), \ldots .$$

The corresponding difference $m - n$ is

$$\ldots, -25, -18, -11, -4, 3, 10, \ldots .$$

As we see, the differences go up by 7. The smallest absolute difference is $\boxed{3}$.

Problem 2 Express $\sqrt{2+\sqrt{3}}$ in the form $\dfrac{a+\sqrt{b}}{\sqrt{c}}$, where a is a positive integer and b and c are square-free positive integers.

Answer: $\dfrac{1+\sqrt{3}}{\sqrt{2}}$ or $\dfrac{3+\sqrt{3}}{\sqrt{6}}$.

Solution: If $\dfrac{a+\sqrt{b}}{\sqrt{c}}$ equals $\sqrt{2+\sqrt{3}}$, then squaring gives

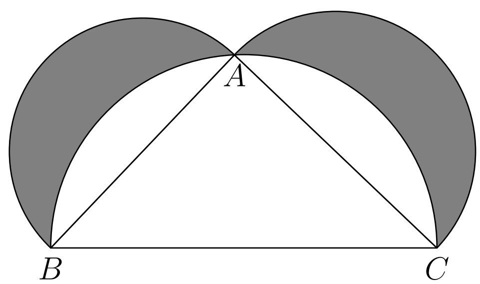
$$\frac{a^2+b+2a\sqrt{b}}{c} = 2+\sqrt{3}.$$

Multiplying by c gives

$$a^2+b+2a\sqrt{b} = 2c + c\sqrt{3}.$$

Rearranging gives
$$a^2 - 2c + b = c\sqrt{3} - 2a\sqrt{b}.$$
To make the square of the right side an integer, $3b$ must be a perfect square. Because b is square-free, we must have $b = 3$. Substituting $b = 3$ gives
$$a^2 - 2c + 3 = (c - 2a)\sqrt{3}.$$
Because $\sqrt{3}$ is irrational, both sides must be zero. So we have $c = 2a$ and $a^2 - 2c + 3 = 0$. Substituting $c = 2a$ into $a^2 - 2c + 3 = 0$ gives
$$a^2 - 4a + 3 = 0.$$
So a is either 1 or 3. The corresponding value of c is 2 or 6, respectively. All the previous steps are reversible, so both solutions are valid.

Problem 3 The figure below shows a triangle ABC with a semicircle on each of its three sides.



If $AB = 20$, $AC = 21$, and $BC = 29$, what is the area of the shaded region?

Answer: 210.

Solution: On one hand, the whole region can be viewed as the (disjoint) union of the semicircle with diameter $\overline{BC}$ and the shaded region. On the other hand, the whole region can be viewed as the (disjoint) union of the semicircle with diameter $\overline{AB}$, the semicircle with diameter $\overline{AC}$, and the triangle ABC. We know that $\angle BAC$ is a right angle, since it is inscribed in a semicircle (or since 20-21-29 is a Pythagorean triple). By the Pythagorean theorem, the area of the semicircle with diameter $\overline{BC}$ must equal the sum of the areas of the two semicircles with diameters $\overline{AB}$ and $\overline{AC}$. The areas of the semicircles cancel out, and hence the area of the shaded region is equal to the area of triangle ABC. The area of the triangle is
$$\frac{20 \cdot 21}{2} = 10 \cdot 21 = \boxed{210}.$$

Note: This problem is basically the ancient problem of finding the area of the lune of Hippocrates.

Problem 4 If $x > 10$, what is the greatest possible value of the expression

$$(\log x)^{\log\log\log x} - (\log\log x)^{\log\log x}?$$

All the logarithms are base 10.

Answer: 0.

Solution: The first term of the given expression is

$$\begin{aligned}(\log x)^{\log\log\log x} &= \left(10^{\log\log x}\right)^{\log\log\log x}\\ &= 10^{(\log\log x)(\log\log\log x)}.\end{aligned}$$

The second term is

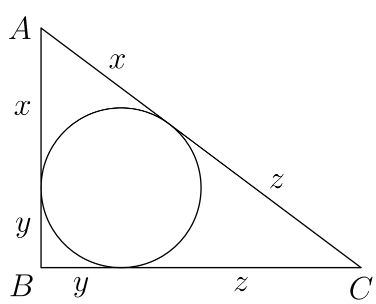
$$\begin{aligned}(\log\log x)^{\log\log x} &= \left(10^{\log\log\log x}\right)^{\log\log x}\\ &= 10^{(\log\log\log x)(\log\log x)}\\ &= 10^{(\log\log x)(\log\log\log x)}.\end{aligned}$$

The two terms are equal! So their difference is always $\boxed{0}$.

Problem 5 Let $\triangle ABC$ be a triangle with $AB = 3$, $BC = 4$, and $AC = 5$. Let I be the center of the circle inscribed in $\triangle ABC$. What is the product of AI, BI, and CI?

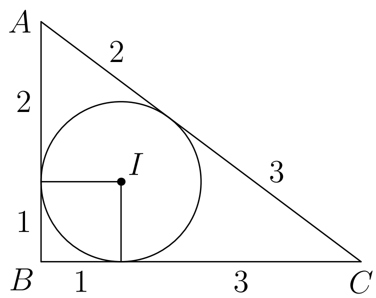
Answer: 10.

Solution: Below is a picture of $\triangle ABC$ and its inscribed circle.



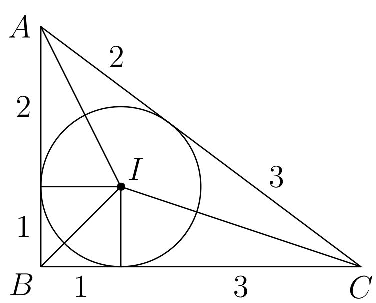
Because the tangents from A to the circle have the same length, we called both lengths x. We did something similar for the other tangents. Because $AB = 3$, we have $x + y = 3$. Similarly $y + z = 4$ and $x + z = 5$. Solving these three equations gives $x = 2$, $y = 1$, and $z = 3$.

Let's subsitute these numbers into our picture. At the same time, let's draw a couple of radii.



Because $\triangle ABC$ is a 3-4-5 triangle, $\angle B$ is a right angle. The two displayed radii form right angles with their tangents. So the displayed quadrilateral is actually a rectangle. In fact, since two of its sides have length 1, the quadrilateral is a square of side length 1. In particular, the radius of the circle is 1.

Next, let's draw segments from the incenter I to the vertices.



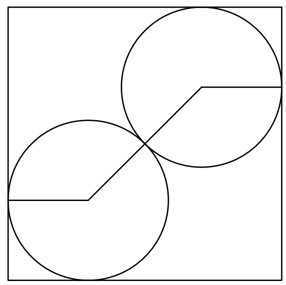
By the Pythagorean theorem, we have $AI = \sqrt{2^2 + 1^2} = \sqrt{5}$. Similarly, we have $BI = \sqrt{1^2 + 1^2} = \sqrt{2}$. Similarly, we have $CI = \sqrt{3^2 + 1^2} = \sqrt{10}$. The product of these three lengths is

$$\sqrt{5} \cdot \sqrt{2} \cdot \sqrt{10} = \sqrt{10} \cdot \sqrt{10} = \boxed{10}.$$

Problem 6 Two circles each have radius 1. No point is inside both circles. The circles are contained in a square. What is the area of the smallest such square? Express your answer in the form $a + b\sqrt{c}$, where a and b are positive integers and c is a square-free positive integer.

Answer: $6 + 4\sqrt{2}$.

Solution: Consider the picture below of two circles in a square.



Each circle has radius 1. The segment joining the two centers forms a 45° angle with the horizontal and vertical lines. The distance between the two centers is 2. So the horizontal distance between the two centers is $\sqrt{2}$. Hence the side length of the square is $2 + \sqrt{2}$. Thus the area of the square is

$$(2 + \sqrt{2})^2 = 4 + 4\sqrt{2} + 2 = 6 + 4\sqrt{2}\,.$$

We will now show that the configuration above is optimal. Consider any configuration of two circles in a square satisfying the conditions of the problem. Let s be the side length of the square. We may impose a coordinate system on the configuration, so that the vertices of the square are $(0, 0)$, $(s, 0)$, (s, s), and $(0, s)$. Because the two circles of radius 1 are contained in the square, the centers of the circles have x-coordinates and y-coordinates between 1 and $s - 1$. So the distance between the two centers is at most $(s - 2)\sqrt{2}$. Because the two circles are disjoint, the distance between the two centers is at least 2. So we get the inequality $(s - 2)\sqrt{2} \geq 2$. Solving for s gives $s \geq 2 + \sqrt{2}$. So the area of the square is at least $(2 + \sqrt{2})^2$, which is $6 + 4\sqrt{2}$.

Problem 7 If z is a complex number such that

$$z + z^{-1} = \sqrt{3},$$

what is the value of

$$z^{2010} + z^{-2010}?$$

Answer: -2.

Solution: We are given that

$$z + z^{-1} = \sqrt{3} = 2 \cdot \frac{\sqrt{3}}{2} = 2\cos\frac{\pi}{6}\,.$$

Hence z is either $\operatorname{cis}\frac{\pi}{6}$ or $\operatorname{cis}(-\frac{\pi}{6})$.

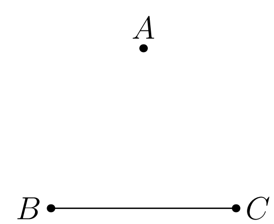
Let's assume that z is $\operatorname{cis}\frac{\pi}{6}$. (The other case is similar.) We have

$$z^{2010} = \left(\operatorname{cis}\frac{\pi}{6}\right)^{2010} = \operatorname{cis}\frac{2010\pi}{6} = \operatorname{cis}(335\pi) = \operatorname{cis}\pi = -1.$$

So z^{-2010} is -1 also. Thus

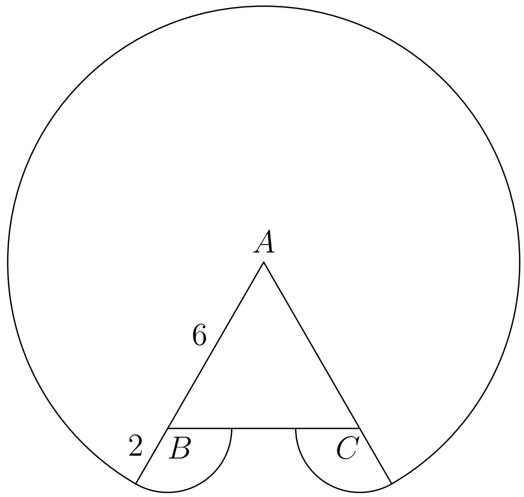
$$z^{2010} + z^{-2010} = -1 + (-1) = \boxed{-2}.$$

Problem 8 In the figure below, points A, B, and C are distance 6 from each other. Say that a point X is *reachable* if there is a path (not necessarily straight) connecting A and X of length at most 8 that does not intersect the interior of $\overline{BC}$. (Both X and the path must lie on the plane containing A, B, and C.) Let R be the set of reachable points. What is the area of R? Express your answer in the form $m\pi + n\sqrt{3}$, where m and n are integers.



Answer: $56\pi + 9\sqrt{3}$.

Solution: The picture below shows all the reachable points.



Each point in $\triangle ABC$ is reachable, since the straight path from that point to A is of length at most 6. Each point in the displayed sector with center A and radius 8 is reachable, since the straight path from that point to A is of length at most 8. Each point in the displayed sector with center B and radius 2 is reachable, since the path from that point to B and then to A is of length at most $2 + 6 = 8$. Similarly, each point in the displayed sector with center C and radius 2 is reachable. No point outside of these regions is reachable.

The sector with center A and radius 8 is a $300°$ sector. Hence the sector is $\frac{5}{6}$ of a circle. Its area is thus $\frac{5}{6}\pi(8)^2$, which simplifies to $\frac{160}{3}\pi$. The sector with center B and radius 2 is a $120°$ sector. Hence the sector is $\frac{1}{3}$ of a circle. Its area is thus $\frac{1}{3}\pi(2)^2$, which simplifies to $\frac{4}{3}\pi$. Similarly, the sector with center C has area $\frac{4}{3}\pi$. The total area of the three sectors is therefore

$$\frac{160\pi}{3} + \frac{4\pi}{3} + \frac{4\pi}{3} = \frac{168\pi}{3} = 56\pi.$$

By the "2-3-4" formula, the area of the equilateral triangle ABC (with side length 6) is

$$\frac{(6)^2\sqrt{3}}{4} = \frac{36\sqrt{3}}{4} = 9\sqrt{3}.$$

So the area of the reachable region is $\boxed{56\pi + 9\sqrt{3}}$.

Problem 9 Let ABC be a triangle. Let D be the midpoint of $\overline{BC}$, let E be the midpoint of $\overline{AD}$, and let F be the midpoint of $\overline{BE}$. Let G be the point where the lines AB and CF intersect. What is the value of $\dfrac{AG}{AB}$? Express your answer as a fraction in simplest form.

Answer: $\dfrac{5}{7}$.

Solution: We will use mass points to solve this problem. Because D is the midpoint of $\overline{BC}$, we have

$$D = \frac{1}{2}B + \frac{1}{2}C.$$

Because E is the midpoint of $\overline{AD}$, we have

$$E = \frac{1}{2}A + \frac{1}{2}D = \frac{1}{2}A + \frac{1}{2}\left(\frac{1}{2}B + \frac{1}{2}C\right) = \frac{1}{2}A + \frac{1}{4}B + \frac{1}{4}C.$$

Because F is the midpoint of $\overline{BE}$, we have

$$F = \frac{1}{2}B + \frac{1}{2}E = \frac{1}{2}B + \frac{1}{2}\left(\frac{1}{2}A + \frac{1}{4}B + \frac{1}{4}C\right) = \frac{1}{4}A + \frac{5}{8}B + \frac{1}{8}C.$$

We can rewrite F as follows:

$$F = \frac{7}{8}\left(\frac{2}{7}A + \frac{5}{7}B\right) + \frac{1}{8}C.$$

Define G' to be $\frac{2}{7}A + \frac{5}{7}B$. By definition, G' is on $\overline{AB}$. On the other hand, from the displayed equation above, we have

$$F = \frac{7}{8}G' + \frac{1}{8}C.$$

So G' is on the line CF. Hence G' must be the point G where lines AB and CF intersect. Looking at the definition of G', we have $AG = \frac{5}{7}AB$. Hence $\frac{AG}{AB}$ is $\frac{5}{7}$.

Problem 10 There are real numbers a and b such that for every positive number x, we have the identity

$$\tan^{-1}\left(\frac{1}{x} - \frac{x}{8}\right) + \tan^{-1}(ax) + \tan^{-1}(bx) = \frac{\pi}{2}\,.$$

(Throughout this equation, $\tan^{-1}$ means the inverse tangent function, sometimes written arctan.) What is the value of $a^2 + b^2$? Express your answer as a fraction in simplest form.

Answer: $\dfrac{3}{4}$.

Solution: Rearranging the given equation, we have

$$\frac{\pi}{2} - \tan^{-1}\big(\frac{1}{x} - \frac{x}{8}\big) = \tan^{-1}(ax) + \tan^{-1}(bx).$$

The tangent of the left side is

$$\tan\left(\frac{\pi}{2} - \tan^{-1}\big(\frac{1}{x} - \frac{x}{8}\big)\right) = \frac{1}{\tan\big(\tan^{-1}\big(\frac{1}{x} - \frac{x}{8}\big)\big)} = \frac{1}{\frac{1}{x} - \frac{x}{8}} = \frac{x}{1 - \frac{1}{8}x^2}\,.$$

By the tangent addition formula, the tangent of the right side is

$$\tan\big(\tan^{-1}(ax) + \tan^{-1}(bx)\big) = \frac{ax + bx}{1 - (ax)(bx)} = \frac{(a+b)x}{1 - abx^2}\,.$$

In order for both sides to be equal for every positive number x, we must have $a + b = 1$ and $ab = \frac{1}{8}$. Hence

$$a^2 + b^2 = (a+b)^2 - 2ab = 1^2 - 2\big(\frac{1}{8}\big) = 1 - \frac{1}{4} = \boxed{\frac{3}{4}}.$$

Problem 11 The sequence a_0, a_1, a_2, $\ldots$ satisfies the recurrence equation

$$a_n = 2a_{n-1} - 2a_{n-2} + a_{n-3}$$

for every integer $n \ge 3$. If $a_{20} = 1$, $a_{25} = 10$, and $a_{30} = 100$, what is the value of a_{1331}?

Answer: 181.

Solution: Let's calculate the first few values of the sequence, looking for a pattern. Plugging $n = 3$ into the recurrence equation, we get

$$a_3 = 2a_2 - 2a_1 + a_0.$$

Plugging in $n = 4$ gives

$$a_4 = 2a_3 - 2a_2 + a_1 = 2(2a_2 - 2a_1 + a_0) = 2a_2 - 3a_1 + 2a_0.$$

Plugging in $n = 5$ gives

$$\begin{aligned} a_5 &= 2a_4 - 2a_3 + a_2 \\ &= 2(2a_2 - 3a_1 + 2a_0) - 2(2a_2 - 2a_1 + a_0) + a_2 \\ &= a_2 - 2a_1 + 2a_0. \end{aligned}$$

Plugging in $n = 6$ gives

$$\begin{aligned} a_6 &= 2a_5 - 2a_4 + a_3 \\ &= 2(a_2 - 2a_1 + 2a_0) - 2(2a_2 - 3a_1 + 2a_0) + (2a_2 - 2a_1 + a_0) \\ &= a_0. \end{aligned}$$

Aha! We discovered that $a_6 = a_0$. A similar calculation reveals that $a_7 = a_1$, $a_8 = a_2$, and so on. The sequence is 6-periodic.

Because the sequence is 6-periodic, we have $a_2 = a_{20} = 1$. Similarly, we have $a_1 = a_{25} = 10$. Also, we have $a_0 = 100$. Finally, we have $a_{1331} = a_5$. According to our work above, we have

$$a_{1331} = a_5 = a_2 - 2a_1 + 2a_0 = 1 - 2(10) + 2(100) = 1 - 20 + 200 = \boxed{181}.$$

Problem 12 If x is a real number, let $\lfloor x \rfloor$ be the greatest integer that is less than or equal to x. If n is a positive integer, let $S(n)$ be defined by

$$S(n) = \left\lfloor \frac{n}{10^{\lfloor \log n \rfloor}} \right\rfloor + 10 \left(n - 10^{\lfloor \log n \rfloor} \cdot \left\lfloor \frac{n}{10^{\lfloor \log n \rfloor}} \right\rfloor \right) .$$

(All the logarithms are base 10.) How many integers n from 1 to 2011 (inclusive) satisfy $S(S(n)) = n$?

Answer: 108.

Solution: The expression $\lfloor \log n \rfloor$ is 1 less than the number of digits of n. So $\left\lfloor \frac{n}{10^{\lfloor \log n \rfloor}} \right\rfloor$ is the first digit (from the left) of n. Hence

$$n - 10^{\lfloor \log n \rfloor} \cdot \left\lfloor \frac{n}{10^{\lfloor \log n \rfloor}} \right\rfloor$$

is just n with the first digit of n erased. Therefore, $S(n)$ itself is a cyclic shift (1 digit to the left) of the digits of n. For example, $S(1234) = 2341$. Also, $S(1034) = 0341 = 341$.

If the second digit of n is nonzero (or if n has just one digit), then $S(S(n))$ is a cyclic shift (2 digits to the left) of the digits of n. For example, $S(S(1234)) = 3412$. But if the second digit of n is zero, then $S(n)$ and hence $S(S(n))$ have fewer digits than n. For example, $S(S(1034)) = 413$.

Now let's count n (with $1 \le n \le 2011$) such that $S(S(n)) = n$. We divide into cases depending on how many digits n has.

Case n is a 1-digit integer (with $n \geq 1$): Then $S(n) = n$, so $S(S(n)) = n$. In other words, all positive 1-digit integers contribute, giving a count of 9.

Case n is a 2-digit integer ab: We have $S(ab) = ba$. If b is nonzero, then $S(S(ab)) = ab$. But if b is zero, then $S(S(ab))$ will have fewer than 2 digits. So we're counting the two-digit integers whose digits are both nonzero. The count is $9 \cdot 9$, or 81.

Case n is a 3-digit integer abc: We have $S(abc) = bca$. Again if b were zero, then $S(S(abc))$ will have too few digits. So b is nonzero. Then $S(S(abc)) = S(bca) = cab$. We need $abc = cab$, which means $a = b = c$. The total count is 9.

Case n is a 4-digit integer $abcd$ (with $n \leq 2011$): We have $S(abcd) = bcda$. Again, we must have b nonzero. Then $S(S(abcd)) = S(bcda) = cdab$. So we must have $abcd = cdab$. Comparing digit by digit, we must have $a = c$ and $b = d$. Because n is at most 2011, we must have $a = c = 1$. We have 9 choices for b. The total count is 9.

Adding all four cases, we get a final count of $9+81+9+9$, which is $\boxed{108}$.

Problem 13 The number 104,060,465 is divisible by a five-digit prime number. What is that prime number?

Answer: 10,613.

Solution: The number 104,060,465 looks like Row 4 of Pascal's triangle: 1, 4, 6, 4, 1. In particular, we have

$$104{,}060{,}465 = 104{,}060{,}401 + 64 = 101^4 + 64 = (101^2)^2 + 64 = 10201^2 + 8^2.$$

Let's try to complete the square:

$$\begin{aligned}
104{,}060{,}465 &= 10201^2 + 8^2 \\
&= \left(10201^2 + 2(10201)8 + 8^2\right)^2 - 2(10201)8 \\
&= (10201 + 8)^2 - 16(10201) \\
&= 10209^2 - 4^2 \cdot 101^2 \\
&= 10209^2 - 404^2.
\end{aligned}$$

We can now apply difference of squares:

$$\begin{aligned}
104{,}060{,}465 &= 10209^2 - 404^2 \\
&= (10209 - 404)(10209 + 404) \\
&= (9805)(10613).
\end{aligned}$$

The five-digit prime divisor of 104,060,465 can't divide 9805, and so must divide 10613. But the only five-digit divisor of 10613 is itself, so the five-digit prime divisor is $\boxed{10613}$.

Here is an alternative solution. We will start with our first observation above:

$$104{,}060{,}465 = 101^4 + 64.$$

We now recall the identity due to Sophie Germain:

$$a^4 + 4b^4 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2).$$

(One way to prove Germain's identity is to add and subtract $4a^2b^2$ on the left side.) Setting $a = 101$ and $b = 2$, we get

$$\begin{aligned}101^4 + 64 &= \left[101^2 + 2(101)2 + 2(2)^2\right] \cdot \left[101^2 - 2(101)2 + 2(2)^2\right] \\ &= (10201 + 404 + 8)(10201 - 404 + 8) \\ &= (9805)(10613).\end{aligned}$$

As before, the five-digit prime number must be $\boxed{10613}$.

Problem 14 If $0 \le p \le 1$ and $0 \le q \le 1$, define $F(p, q)$ by

$$F(p, q) = -2pq + 3p(1 - q) + 3(1 - p)q - 4(1 - p)(1 - q).$$

Define $G(p)$ to be the maximum of $F(p, q)$ over all q (in the interval $0 \le q \le 1$). What is the value of p (in the interval $0 \le p \le 1$) that minimizes $G(p)$? Express your answer as a fraction in simplest form.

Answer: $\dfrac{7}{12}$.

Solution: For fixed p, the expression for $F(p, q)$ is linear in q. Hence $G(p)$ is the maximum of the end values $F(p, 0)$ and $F(p, 1)$. Plugging in $q = 0$, we see that $F(p, 0)$ is

$$F(p, 0) = 3p - 4(1 - p) = 7p - 4.$$

Plugging in $q = 1$, we see that $F(p, 1)$ is

$$F(p, 1) = -2p + 3(1 - p) = 3 - 5p.$$

Hence $G(p)$ is $\max(7p - 4, 3 - 5p)$.

To get a feel for $G(p)$, let's compare the two expressions $7p - 4$ and $3 - 5p$. When $p = \frac{7}{12}$, the two expressions are equal. When $p < \frac{7}{12}$, the second expression is greater. When $p > \frac{7}{12}$, the first expression is greater. So G is linear on $[0, \frac{7}{12}]$ and is linear on $[\frac{7}{12}, 1]$. We calculate that $G(0) = 3$, $G(\frac{7}{12}) = \frac{1}{12}$, and $G(1) = 3$. So the minimum value of $G(p)$ occurs when p is $\boxed{\dfrac{7}{12}}$.

Problem 15 The game of backgammon has a "doubling" cube, which is like a standard 6-faced die except that its faces are inscribed with the numbers 2, 4, 8, 16, 32, and 64, respectively. After rolling the doubling cube four times at random, we let a be the value of the first roll, b be the value of the second roll, c be the value of the third roll, and d be the value of the fourth roll. What is the probability that $\dfrac{a+b}{c+d}$ is the average of $\dfrac{a}{c}$ and $\dfrac{b}{d}$? Express your answer as a fraction in simplest form.

Answer: $\dfrac{163}{648}$.

Solution: The average of $\frac{a}{c}$ and $\frac{b}{d}$ is

$$\frac{1}{2}\left(\frac{a}{c} + \frac{b}{d}\right) = \frac{ad+bc}{2cd}\,.$$

So for $\frac{a+b}{c+d}$ to equal this average means

$$\frac{a+b}{c+d} = \frac{ad+bc}{2cd}\,.$$

Clearing fractions gives

$$2(a+b)cd = (ad+bc)(c+d) = acd + ad^2 + bc^2 + bcd.$$

Putting all the terms on one side gives

$$ad^2 - acd - bcd + bc^2 = 0.$$

Factoring gives

$$(ad - bc)(d - c) = 0.$$

All the steps above are reversible. So $\frac{a+b}{c+d}$ is the average of $\frac{a}{c}$ and $\frac{b}{d}$ if and only if $ad = bc$ or $c = d$.

Let's compute separately the probability that $ad = bc$ and the probability that $c = d$. First, the probability that $c = d$ is $\frac{1}{6}$.

Let's now compute the probability that $ad = bc$. The expression ad has the possible values

$$2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}, 2^{11}, 2^{12}$$

with respective probabilities

$$\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}.$$

The same is true of the expression bc. So the probability that $ad = bc$ is

$$\begin{aligned}
\Pr(ad = bc) &= \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2}{36^2} \\
&= \frac{2(1^2 + 2^2 + 3^2 + 4^2 + 5^2) + 6^2}{1296} \\
&= \frac{2(1 + 4 + 9 + 16 + 25) + 36}{1296} \\
&= \frac{2(55) + 36}{1296} \\
&= \frac{146}{1296} \\
&= \frac{73}{648}.
\end{aligned}$$

To avoid double-counting, we also have to compute the probability that $ad = bc$ and $c = d$. The only way for that to happen is if $a = b$ and $c = d$. So the probability is $\frac{1}{6} \cdot \frac{1}{6}$, which is $\frac{1}{36}$.

By the Principle of Inclusion-Exclusion, the probability that $ad = bc$ or $c = d$ is

$$\frac{73}{648} + \frac{1}{6} - \frac{1}{36} = \frac{73}{648} + \frac{108}{648} - \frac{18}{648} = \frac{73 + 108 - 18}{648} = \boxed{\frac{163}{648}}.$$

Problem 16 Let N be the number of ordered pairs of integers (x, y) such that

$$4x^2 + 9y^2 \le 1000000000.$$

Let a be the first digit of N (from the left) and let b be the second digit of N. What is the value of $10a + b$?

Answer: 52.

Solution: The value of N is the number of lattice points (points with integer coordinates) inside the ellipse $(2x)^2 + (3y)^2 \le 1{,}000{,}000{,}000$. Imagine a unit square at the center of each such lattice point. Then N is the number of such unit squares. Equivalently, N is the area of the union of all such unit squares. The union of such unit squares is close in area to the ellipse. The area of the ellipse is $\frac{1{,}000{,}000{,}000}{6}\pi$, which is approximately

$$\frac{1{,}000{,}000{,}000}{6}(3.1416) = \frac{3{,}141{,}600{,}000}{6} = 523{,}600{,}000.$$

So N is approximately 523,600,000. Assuming the approximation is close enough, that means a is 5 and b is 2. So $10a + b$ is 52.

In the rest of the solution, we will prove that the approximation is close enough. Imagine stretching the entire picture by a factor of 2 in the x direction and by a factor of 3 in the y direction. The ellipse transforms into a circle with radius $\sqrt{10^9}$. Each unit square transforms into a 2-by-3 rectangle. The union of these N rectangles has area $6N$. In each such rectangle, the distance of every point in the rectangle to the center of the rectangle is at most

$$\sqrt{1^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{13}{4}}\,.$$

So the union of the rectangles is contained in a big circle of radius $\sqrt{10^9} + \sqrt{\frac{13}{4}}$. Similarly, the union of the rectangles contains a small circle of radius $\sqrt{10^9} - \sqrt{\frac{13}{4}}$.

The area of the big circle is

$$\pi\left(\sqrt{10^9} + \sqrt{\frac{13}{4}}\right)^2 = \pi\left(10^9 + 10^4\sqrt{130} + \frac{13}{4}\right) < 10^9\pi + 4\cdot 10^4\cdot 12 + 13 < 3{,}150{,}000.$$

The area of the small circle is

$$\pi\left(\sqrt{10^9} - \sqrt{\frac{13}{4}}\right)^2 = \pi\left(10^9 - 10^4\sqrt{130} + \frac{13}{4}\right) > 10^9\pi - 4\cdot 10^4\cdot 12 > 3{,}120{,}000.$$

That means $6N$, the area of the union of the rectangles, is between 3,120,000 and 3,150,000. So N itself is between 520,000 and 525,000. Thus our approximation is close enough.

Problem 17 There is a polynomial P such that for every real number x,

$$x^{512} + x^{256} + 1 = (x^2 + x + 1)P(x).$$

When P is written in standard polynomial form, how many of its coefficients are nonzero?

Answer: 341.

Solution: We can cleverly rewrite $x^{512} + x^{256} + 1$ as follows:

$$\begin{aligned} x^{512} + x^{256} + 1 = (x^{512} + x^{511} + \cdots + 1) \\ - (x^{511} + x^{510} + \cdots + x^{257}) \\ - (x^{255} + x^{254} + \cdots + x). \end{aligned}$$

Each of the three parenthesized polynomials is a multiple of $x^2 + x + 1$. For example,

$$x^{512} + x^{511} + \cdots + 1 = (x^2 + x + 1)(x^{510} + x^{507} + \cdots + 1).$$

(The exponents in the rightmost factor skip down by 3. So that factor has 171 terms.) Similarly, we have

$$x^{511} + x^{510} + \cdots + x^{257} = (x^2 + x + 1)(x^{509} + x^{506} + \cdots + x^{257}).$$

(The rightmost factor has 85 terms.) Finally, we have

$$x^{255} + x^{254} + \cdots + x = (x^2 + x + 1)(x^{253} + x^{250} + \cdots + x).$$

(The rightmost factor has 85 terms.) By subtracting the last two equations from the equation before that, we get a factorization of $x^{512} + x^{256} + 1$ as the product of $x^2 + x + 1$ and a polynomial with $171 + 85 + 85 = 341$ terms. The terms all have different exponents. So the answer to the problem is $\boxed{341}$.

Problem 18 The polynomial P is a quadratic with integer coefficients. For every positive integer n, the integers $P(n)$ and $P(P(n))$ are relatively prime to n. If $P(3) = 89$, what is the value of $P(10)$?

Answer: 859.

Solution: Throughout we will use the fact that if $a \equiv b \pmod{n}$, then $P(a) \equiv P(b) \pmod{n}$.

First, we will determine $P(0)$. For every positive integer n, we have

$$0 \equiv n \pmod{n}.$$

Applying P on both sides, we have

$$P(0) \equiv P(n) \pmod{n}.$$

Because $P(n)$ and n are relatively prime, $P(0)$ and n are relatively prime. Since this is true for every positive integer n (and in particular, for every prime), $P(0)$ is either -1 or 1. To determine which one, we work mod 3:

$$P(0) \equiv P(3) = 89 \equiv 2 \pmod{3}.$$

Hence $P(0) = -1$.

Next, we will determine $P(-1)$. Recall the equation

$$P(0) \equiv P(n) \pmod{n}.$$

Because $P(0) = -1$, we have

$$-1 \equiv P(n) \pmod{n}.$$

Applying P on both sides, we have

$$P(-1) \equiv P(P(n)) \pmod{n}.$$

Because $P(P(n))$ and n are relatively prime, $P(-1)$ and n are relatively prime. As before, $P(-1)$ must be either -1 or 1. To determine which one, we use the fact that

$$-1 \equiv 3 \pmod{4}.$$

Applying P on both sides, we get

$$P(-1) \equiv P(3) = 89 \equiv 1 \pmod{4}.$$

Hence $P(-1)$ is 1.

We have discovered that $P(0) = -1$, $P(-1) = 1$, and $P(3) = 89$. There is a unique quadratic that satisfies all three equations. One way to find this quadratic is to write $P(x)$ as follows:

$$P(x) = a + bx + cx(x+1).$$

Plugging in $x = 0$, we find $a = -1$. Plugging in $x = -1$, we find $b = -2$. Plugging in $x = 3$, we find $c = 8$. So the polynomial P is

$$P(x) = a + bx + cx(x+1) = -1 - 2x + 8x(x+1) = 8x^2 + 6x - 1.$$

In particular, we have

$$P(10) = 8(10)^2 + 6(10) - 1 = 800 + 60 - 1 = \boxed{859}.$$

Problem 19 If $-1 < x < 1$ and $-1 < y < 1$, define the "relativistic sum" $x \oplus y$ to be

$$x \oplus y = \frac{x+y}{1+xy}\,.$$

The operation $\oplus$ is commutative and associative. Let v be the number

$$v = \frac{\sqrt[7]{17} - 1}{\sqrt[7]{17} + 1}\,.$$

What is the value of

$$v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v\,?$$

(In this expression, $\oplus$ appears 13 times.) Express your answer as a fraction in simplest form.

Answer: $\dfrac{144}{145}$.

Solution: The number v is of the form $\frac{a-1}{a+1}$, where $a = \sqrt[7]{17}$. If b and c are positive numbers, then

$$\begin{aligned}
\frac{b-1}{b+1} \oplus \frac{c-1}{c+1} &= \frac{\frac{b-1}{b+1} + \frac{c-1}{c+1}}{1 + \frac{b-1}{b+1} \cdot \frac{c-1}{c+1}} \\
&= \frac{(b-1)(c+1) + (b+1)(c-1)}{(b+1)(c+1) + (b-1)(c-1)} \\
&= \frac{bc - c + b - 1 + bc + c - b - 1}{bc + b + c + 1 + bc - b - c + 1} \\
&= \frac{2bc - 2}{2bc + 2} \\
&= \frac{bc - 1}{bc + 1}\,.
\end{aligned}$$

In other words, if we define $f(t) = \frac{t-1}{t+1}$ for any positive number t, then $f(b) \oplus f(c) = f(bc)$. We can extend this equation to more terms. For example, if b, c, and d are positive, then

$$f(b) \oplus f(c) \oplus f(d) = f(bc) \oplus f(d) = f(bcd).$$

(This reasoning also explains why $\oplus$ is associative.) And so on.

Going back to our original problem, we have

$$\begin{aligned}
\underbrace{v \oplus v \oplus \cdots \oplus v}_{\text{14 copies of } v} &= \underbrace{f(a) \oplus f(a) \oplus \cdots \oplus f(a)}_{\text{14 copies of } f(a)} \\
&= f(\underbrace{a \times a \times \cdots \times a}_{\text{14 copies of } a}) \\
&= f(a^{14}) \\
&= f(17^2) \\
&= f(289) \\
&= \frac{289-1}{289+1} \\
&= \frac{288}{290} \\
&= \boxed{\frac{144}{145}}.
\end{aligned}$$

Note: Our "relativistic sum" definition was inspired by the addition law for velocities in special relativity.

Problem 20 Let ABC be an equilateral triangle with each side of length 1. Let X be a point chosen uniformly at random on side $\overline{AB}$. Let Y be a point chosen uniformly at random on side $\overline{AC}$. (Points X and Y are chosen independently.) Let p be the probability that the distance XY is at most $\dfrac{1}{\sqrt[4]{3}}$. What is the value of $900p$, rounded to the nearest integer?

Answer: 628.

Solution: Let $x = AX$ and $y = AY$. By the Law of Cosines, the distance squared between X and Y is

$$x^2 + y^2 - 2xy\cos 60^\circ = x^2 + y^2 - 2xy \cdot \frac{1}{2} = x^2 + y^2 - xy.$$

So $XY \le \frac{1}{\sqrt[4]{3}}$ is equivalent to

$$x^2 + y^2 - xy \le \frac{1}{\sqrt{3}} \, .$$

The graph of this inequality is an ellipse, whose axes are rotated 45° from the coordinate axes. If $XY \le \frac{1}{\sqrt[4]{3}}$, then $XY < \frac{\sqrt{3}}{2}$, so X can't be the vertex C, which means that x can't be 1. As a consequence, the ellipse doesn't intersect the line $x = 1$. Similarly, the ellipse doesn't intersect the line $y = 1$. So the portion of the ellipse in the first quadrant lies in the unit square. The area of that portion of the ellipse is the value of p.

The ellipse inequality

$$x^2 + y^2 - xy \le \frac{1}{\sqrt{3}}$$

is equivalent to the inequality

$$\left(\frac{x+y}{\sqrt{2}}\right)^2 + 3\left(\frac{x-y}{\sqrt{2}}\right)^2 \le \frac{2}{\sqrt{3}} \, .$$

Let's rotate the entire graph by 45° clockwise. Then the original ellipse transforms to the axes-parallel ellipse

$$x^2 + 3y^2 \le \frac{2}{\sqrt{3}} \, .$$

The boundary lines $x = 0$ and $y = 0$ transform to the lines $y = x$ and $y = -x$. Now let's transform the new ellipse to a circle by stretching the entire figure in the y direction by a factor of $\sqrt{3}$. The axes-parallel ellipse transforms to the circle

$$x^2 + y^2 \le \frac{2}{\sqrt{3}} \, .$$

The original boundary lines $y = x$ and $y = -x$ transform to the lines $y = \sqrt{3}x$ and $y = -\sqrt{3}x$. These new boundary lines form an angle of 120°. So the area of the circle sector inside the boundary lines is

$$\frac{1}{3} \cdot \pi \cdot \frac{2}{\sqrt{3}} = \frac{2\pi}{3\sqrt{3}} \, .$$

Because of the stretch, the area of the portion of the ellipse inside the boundary lines is

$$\frac{1}{\sqrt{3}} \cdot \frac{2\pi}{3\sqrt{3}} = \frac{2\pi}{9} \, .$$

Thus $p = \frac{2\pi}{9}$. Hence $900p$ is 200π, which rounds to 628.