The Advantage Testing Foundation

Math Prize for Girls at MIT

2012 Olympiad Solutions

Problem 1 Let $A_1A_2\dots A_n$ be a polygon (not necessarily regular) with n sides. Suppose there is a translation that maps each point A_i to a point B_i in the same plane. For convenience, define $A_0 = A_n$ and $B_0 = B_n$. Prove that

$$\sum_{i=1}^{n}(A_{i-1}B_i)^2 = \sum_{i=1}^{n}(B_{i-1}A_i)^2 \,.$$

Solution: We will use vectors to solve this problem. By definition of translation, there is a vector v such that $B_i = A_i + v$ for each i. Let's analyze the left-hand side of the desired equation:

$$\begin{aligned}
\sum_{i=1}^{n}(A_{i-1}B_i)^2 &= \sum_{i=1}^{n}\left\|B_i - A_{i-1}\right\|^2 \\
&= \sum_{i=1}^{n}\left\|A_i - A_{i-1} + v\right\|^2 \\
&= \sum_{i=1}^{n}\left(\left\|A_i - A_{i-1}\right\|^2 + 2(A_i - A_{i-1})\cdot v + \left\|v\right\|^2\right) \\
&= \sum_{i=1}^{n}\left\|A_i - A_{i-1}\right\|^2 + 2v\cdot\sum_{i=1}^{n}(A_i - A_{i-1}) + \sum_{i=1}^{n}\left\|v\right\|^2 \\
&= \sum_{i=1}^{n}(A_{i-1}A_i)^2 + 2v\cdot 0 + n\left\|v\right\|^2 \\
&= \sum_{i=1}^{n}(A_{i-1}A_i)^2 + n\left\|v\right\|^2 \,.
\end{aligned}$$

Similarly, let's analyze the right-hand side of the desired equation:

$$\begin{aligned}
\sum_{i=1}^{n}(B_{i-1}A_i)^2 &= \sum_{i=1}^{n} \|A_i - B_{i-1}\|^2 \\
&= \sum_{i=1}^{n} \|A_i - A_{i-1} - v\|^2 \\
&= \sum_{i=1}^{n} \left(\|A_i - A_{i-1}\|^2 - 2(A_i - A_{i-1}) \cdot v + \|v\|^2\right) \\
&= \sum_{i=1}^{n} \|A_i - A_{i-1}\|^2 - 2v \cdot \sum_{i=1}^{n}(A_i - A_{i-1}) + \sum_{i=1}^{n} \|v\|^2 \\
&= \sum_{i=1}^{n}(A_{i-1}A_i)^2 - 2v \cdot 0 + n\,\|v\|^2 \\
&= \sum_{i=1}^{n}(A_{i-1}A_i)^2 + n\,\|v\|^2 .
\end{aligned}$$

We see that the left-hand side and right-hand side of the desired equation are equal to the same expression, and so must equal each other.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 2 Let m and n be integers greater than 1. Prove that $\left\lfloor \dfrac{mn}{6} \right\rfloor$ non-overlapping 2-by-3 rectangles can be placed in an m-by-n rectangle. Note: $\lfloor x \rfloor$ means the greatest integer that is less than or equal to x.

Solution: Say that (m, n) is *good* if we can pack $\lfloor mn/6 \rfloor$ disjoint 2-by-3 rectangles in an m-by-n rectangle. (In other words, the wasted space has area less than 6.) Say that (m, n) is *perfect* if we can pack $mn/6$ disjoint 2-by-3 rectangles in an m-by-n rectangle. (In other words, there is no wasted space.)

It's easy to check that if m is even and n is a multiple of 3 (or vice versa), then (m, n) is perfect.

In particular, $(2, 6)$ and $(3, 6)$ are perfect. Every integer $m > 1$ can be written as the sum of 2's and 3's. So $(m, 6)$ is always perfect.

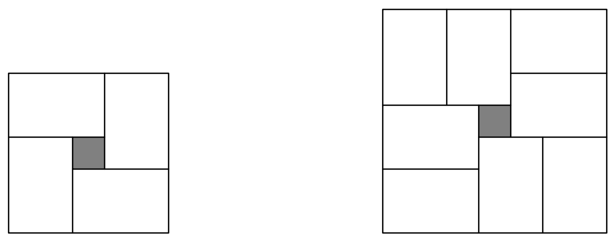
Assume by strong induction that the result is true for every rectangle with area less than mn. We will prove that (m, n) is good. Without loss of generality, we may assume that $m \le n$.

Suppose that $n \geq 8$. Then $(m, 6)$ is perfect and $(m, n-6)$ is good by induction. So (m, n) is good.

Suppose that m is even and $n \geq 5$. Then $(m, 3)$ is perfect and $(m, n-3)$ is good by induction. So (m, n) is good.

Suppose that m is a multiple of 3 and $n \geq 4$. Then $(m, 2)$ is perfect and $(m, n-2)$ is good by induction. So (m, n) is good.

Only 9 cases remain: $(2, 2)$, $(2, 3)$, $(2, 4)$, $(3, 3)$, $(4, 4)$, $(5, 5)$, $(5, 6)$, $(5, 7)$, and $(7, 7)$. Each of these cases is straightforward to check, except perhaps for $(5, 5)$ and $(7, 7)$. Below are good packings for $(5, 5)$ and $(7, 7)$.



Problem 3 Recall that the *Fibonacci numbers* are defined recursively by the equation

$$F_n = F_{n-1} + F_{n-2}$$

for every integer $n \geq 2$, with initial values $F_0 = 0$ and $F_1 = 1$. Let k be a positive integer. Say that an integer is *k-summable* if it is the sum of k Fibonacci numbers (not necessarily distinct).

(a) Prove that every positive integer less than $F_{2k+3} - 1$ is k-summable.

(b) Prove that $F_{2k+3} - 1$ is not k-summable.

Solution:

(a) For every nonnegative integer k, we will prove that every nonnegative integer less than $F_{2k+3} - 1$ is k-summable. The proof is by induction on k. The base case $k = 0$ is trivial. Assume by induction that the result is true for $k - 1$, where $k \geq 1$. Let n be a nonnegative integer less than $F_{2k+3} - 1$. We will show that n is k-summable.

Let j be the nonnegative integer such that $F_j \leq n < F_{j+1}$. We will show that $n - F_j$ is less than $F_{2k+1} - 1$. If $j \geq 2k + 2$, then

$$n - F_j \leq n - F_{2k+2} < (F_{2k+3} - 1) - F_{2k+2} = F_{2k+1} - 1.$$

On the other hand, if $j \leq 2k+1$, then

$$n - F_j < F_{j+1} - F_j = F_{j-1} \leq F_{2k} = F_{2k+1} - F_{2k-1} \leq F_{2k+1} - 1.$$

Either way, $n - F_j$ is less than $F_{2k+1} - 1$. By induction, $n - F_j$ is $(k-1)$-summable. Because n is the sum of $n - F_j$ and F_j, it follows that n is k-summable. We have completed the proof by induction.

(b) Among all the representations of $F_{2k+3} - 1$ as the sum of Fibonacci numbers, consider the one(s) with the fewest number of terms. Among these representations, consider one with the smallest sum of Fibonacci indices.

We claim that this representation doesn't contain two consecutive Fibonacci numbers. That's because $F_j + F_{j+1}$ could be replaced by F_{j+2}, which reduces the number of terms.

We claim that this representation doesn't contain the same term twice. We could replace $F_0 + F_0$ by F_0, which reduces the number of terms. We could replace $F_1 + F_1$ by F_3, which reduces the number of terms. For $j \geq 2$, we could replace $F_j + F_j$ by $F_{j+1} + F_{j-2}$. The number of terms has stayed the same, and so has the sum of the terms, because

$$F_j + F_j = F_j + (F_{j-1} + F_{j-2}) = F_{j+1} + F_{j-2}\,.$$

But the sum of the Fibonacci indices has gone down, because $(j+1)+(j-2)$ is less than $j+j$.

So this representation expresses $F_{2k+3}-1$ as the sum of Fibonacci numbers with distinct, non-consecutive indices. The largest term is at most F_{2k+2}. Because the indices are distinct and non-consecutive, the second largest term is at most F_{2k}. And so on. At this point, we will need the following sum of even-indexed Fibonacci numbers:

$$\begin{aligned}
F_4 + F_6 + \cdots + F_{2k+2} &= (F_5 - F_3) + (F_7 - F_5) + \cdots + (F_{2k+3} - F_{2k+1}) \\
&= F_{2k+3} - F_3 \\
&= F_{2k+3} - 2 \\
&< F_{2k+3} - 1.
\end{aligned}$$

The sum on the left has k terms. So the sum of the k largest terms in our representation is less than $F_{2k+3} - 1$. Hence our representation of $F_{2k+3} - 1$ has more than k terms. Because our representation has the fewest number of terms, every representation of $F_{2k+3} - 1$ as the sum of Fibonacci numbers has more than k terms. So $F_{2k+3} - 1$ is not k-summable.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 4 Let f be a function from the set of rational numbers to the set of real numbers. Suppose that for all rational numbers r and s, the expression $f(r+s) - f(r) - f(s)$ is an integer. Prove that there is a positive integer q and an integer p such that

$$\left| f\left(\frac{1}{q}\right) - p \right| \le \frac{1}{2012}.$$

Solution: We will use Dirichlet's approximation theorem. For convenience, we state and prove it below.

Theorem 1 (Dirichlet's Approximation Theorem) *If x is a real number, and n is a positive integer, then there are integers a and b such that $1 \le a \le n$ and*

$$|ax - b| \le \frac{1}{n+1}. \qquad \square$$

PROOF We need to show that there is a positive integer $a \le n$ such that ax is within $\frac{1}{n+1}$ of an integer. Divide the unit interval $[0,1)$ into $n+1$ equal intervals: $[0, \frac{1}{n+1})$, $[\frac{1}{n+1}, \frac{2}{n+1})$, $[\frac{2}{n+1}, \frac{3}{n+1})$, …, $[\frac{n}{n+1}, 1)$. Consider the sequence of numbers x, $2x$, …, nx, modulo 1. If one of them (say $ix \bmod 1$) is in either of the extreme intervals $[0, \frac{1}{n+1})$ or $[\frac{n}{n+1}, 1)$, then we're done (choose $a = i$). Otherwise, we have n numbers in the $n-1$ inner intervals. By the Pigeonhole Principle, two of the numbers (say $ix \bmod 1$ and $jx \bmod 1$) are in the same interval. Then $|ix - jx|$ is less than $\frac{1}{n+1}$ from an integer. By choosing $a = |i-j|$, we're done. ■

Now back to the solution of the problem. Let N be a common multiple of 1, 2, …, 2011. (For example, $N = 2011!$.) By Dirichlet's approximation theorem (with $x = f(1/N)$ and $n = 2011$), there is a positive integer $a \le 2011$ and an integer b such that

$$\left| af\left(\frac{1}{N}\right) - b \right| \le \frac{1}{2012}.$$

By the main condition of the problem, $f(a/N)$ and $af(1/N)$ differ by an integer. So there is an integer p such that

$$\left| f\left(\frac{a}{N}\right) - p \right| \le \frac{1}{2012}.$$

Because N is a multiple of a, we can consider the integer $q = N/a$. Then

$$\left| f\left(\frac{1}{q}\right) - p \right| \leq \frac{1}{2012}\,.$$

So we're done.