

# MATH PRIZE FOR GIRLS OLYMPIAD

WEDNESDAY, NOVEMBER 7, 2012

TIME LIMIT: 4 HOURS

1. Let  $A_1A_2 \dots A_n$  be a polygon (not necessarily regular) with  $n$  sides. Suppose there is a translation that maps each point  $A_i$  to a point  $B_i$  in the same plane. For convenience, define  $A_0 = A_n$  and  $B_0 = B_n$ . Prove that

$$\sum_{i=1}^n (A_{i-1}B_i)^2 = \sum_{i=1}^n (B_{i-1}A_i)^2.$$

2. Let  $m$  and  $n$  be integers greater than 1. Prove that  $\lfloor \frac{mn}{6} \rfloor$  non-overlapping 2-by-3 rectangles can be placed in an  $m$ -by- $n$  rectangle. Note:  $\lfloor x \rfloor$  means the greatest integer that is less than or equal to  $x$ .
3. Recall that the *Fibonacci numbers* are defined recursively by the equation

$$F_n = F_{n-1} + F_{n-2}$$

for every integer  $n \geq 2$ , with initial values  $F_0 = 0$  and  $F_1 = 1$ . Let  $k$  be a positive integer. Say that an integer is *k-summable* if it is the sum of  $k$  Fibonacci numbers (not necessarily distinct).

- (a) Prove that every positive integer less than  $F_{2k+3} - 1$  is  $k$ -summable.
- (b) Prove that  $F_{2k+3} - 1$  is not  $k$ -summable.
4. Let  $f$  be a function from the set of rational numbers to the set of real numbers. Suppose that for all rational numbers  $r$  and  $s$ , the expression  $f(r+s) - f(r) - f(s)$  is an integer. Prove that there is a positive integer  $q$  and an integer  $p$  such that

$$\left| f\left(\frac{1}{q}\right) - p \right| \leq \frac{1}{2012}.$$