Math Prize for Girls Olympiad

Wednesday, November 7, 2012

Time Limit: 4 hours

1. Let $A_1 A_2 \ldots A_n$ be a polygon (not necessarily regular) with n sides. Suppose there is a translation that maps each point A_i to a point B_i in the same plane. For convenience, define $A_0 = A_n$ and $B_0 = B_n$. Prove that

$$\sum_{i=1}^{n} (A_{i-1}B_i)^2 = \sum_{i=1}^{n} (B_{i-1}A_i)^2 \,.$$

2. Let m and n be integers greater than 1. Prove that $\left\lfloor \dfrac{mn}{6} \right\rfloor$ non-overlapping 2-by-3 rectangles can be placed in an m-by-n rectangle. Note: $\lfloor x \rfloor$ means the greatest integer that is less than or equal to x.

3. Recall that the *Fibonacci numbers* are defined recursively by the equation

$$F_n = F_{n-1} + F_{n-2}$$

for every integer $n \ge 2$, with initial values $F_0 = 0$ and $F_1 = 1$. Let k be a positive integer. Say that an integer is k-*summable* if it is the sum of k Fibonacci numbers (not necessarily distinct).

(a) Prove that every positive integer less than $F_{2k+3} - 1$ is k-summable.

(b) Prove that $F_{2k+3} - 1$ is not k-summable.

4. Let f be a function from the set of rational numbers to the set of real numbers. Suppose that for all rational numbers r and s, the expression $f(r+s) - f(r) - f(s)$ is an integer. Prove that there is a positive integer q and an integer p such that

$$\left| f\left(\frac{1}{q}\right) - p \right| \le \frac{1}{2012} \,.$$