Advantage Testing Foundation

The Fourth Annual Math Prize for Girls

Saturday, September 22, 2012

TEST BOOKLET

Test Version A

DIRECTIONS

1. Do not open this test until your proctor instructs you to.
2. Fill out the top of your answer sheet. Your student number is on your name tag. Your test version (A, B, C, or D) is above on this page.
3. You may use pencils, pens, and erasers. You may also use the scratch paper that we provide. You may NOT use calculators, rulers, protractors, compasses, graph paper, books, or slide rules.
4. Write your final answers on the answer sheet. When a problem specifies a particular form for the answer, write your answer in that form. Write your answers clearly. We will collect only the answer sheet from you.

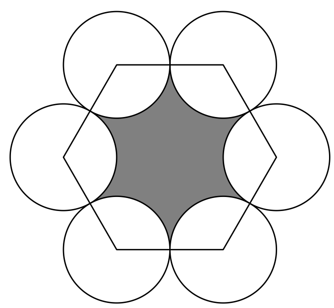
NOTES

1. This test contains 20 problems. You will have 150 minutes (2.5 hours) to take the test. Your score will be the number of correct answers.
2. Figures are not necessarily drawn to scale.
3. Good luck!

Math Prize for Girls 2012

1. In the morning, Esther biked from home to school at an average speed of x miles per hour. In the afternoon, having lent her bike to a friend, Esther walked back home along the same route at an average speed of 3 miles per hour. Her average speed for the round trip was 5 miles per hour. What is the value of x?

2. In the figure below, the centers of the six congruent circles form a regular hexagon with side length 2.



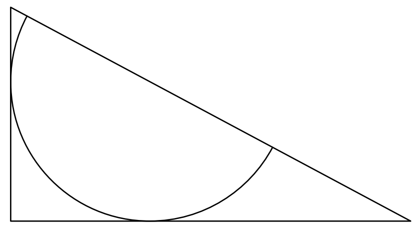
Adjacent circles are tangent to each other. What is the area of the shaded region? Express your answer in the form $a\sqrt{n} - b\pi$, where a and b are positive integers and n is a square-free positive integer.

3. What is the least positive integer n such that $n!$ is a multiple of 2012^{2012}?

4. Evaluate the expression

$$\frac{121\left(\frac{1}{13} - \frac{1}{17}\right) + 169\left(\frac{1}{17} - \frac{1}{11}\right) + 289\left(\frac{1}{11} - \frac{1}{13}\right)}{11\left(\frac{1}{13} - \frac{1}{17}\right) + 13\left(\frac{1}{17} - \frac{1}{11}\right) + 17\left(\frac{1}{11} - \frac{1}{13}\right)}.$$

5. The figure below shows a semicircle inscribed in a right triangle.



The triangle has legs of length 8 and 15. The semicircle is tangent to the two legs, and its diameter is on the hypotenuse. What is the radius of the semicircle? Express your answer as a fraction in simplest form.

6. For how many ordered pairs of positive integers (x, y) is the least common multiple of x and y equal to 1,003,003,001?

7. Let f_1, f_2, f_3, $\ldots$, be a sequence of numbers such that

$$f_n = f_{n-1} + f_{n-2}$$

for every integer $n \geq 3$. If $f_7 = 83$, what is the sum of the first 10 terms of the sequence?

8. Suppose that x, y, and z are real numbers such that $x + y + z = 3$ and $x^2 + y^2 + z^2 = 6$. What is the largest possible value of z? Express your answer in the form $a + \sqrt{b}$, where a and b are positive integers.

9. Bianca has a rectangle whose length and width are distinct primes less than 100. Let P be the perimeter of her rectangle, and let A be the area of her rectangle. What is the least possible value of $\frac{P^2}{A}$? Express your answer as a fraction in simplest form.

10. Let $\triangle ABC$ be a triangle with a right angle $\angle ABC$. Let D be the midpoint of $\overline{BC}$, let E be the midpoint of $\overline{AC}$, and let F be the midpoint of $\overline{AB}$. Let G be the midpoint of $\overline{EC}$. One of the angles of $\triangle DFG$ is a right angle. What is the least possible value of $\frac{BC}{AG}$? Express your answer as a fraction in simplest form.

11. Alison has an analog clock whose hands have the following lengths: a inches (the hour hand), b inches (the minute hand), and c inches (the second hand), with $a < b < c$. The numbers a, b, and c are consecutive terms of an arithmetic sequence. The tips of the hands travel the following distances during a day: A inches (the hour hand), B inches (the minute hand), and C inches (the second hand). The numbers A, B, and C (in this order) are consecutive terms of a geometric sequence. What is the value of $\frac{B}{A}$? Express your answer in the form $x + y\sqrt{n}$, where x and y are positive integers and n is a square-free positive integer.

12. What is the sum of all positive integer values of n that satisfy the equation

$$\cos\left(\frac{\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right)\cos\left(\frac{4\pi}{n}\right)\cos\left(\frac{8\pi}{n}\right)\cos\left(\frac{16\pi}{n}\right) = \frac{1}{32}\,?$$

13. For how many integers n with $1 \le n \le 2012$ is the product

$$\prod_{k=0}^{n-1} \left(\left(1 + e^{2\pi ik/n}\right)^n + 1 \right)$$

equal to zero?

14. Let k be the smallest positive integer such that the binomial coefficient $\binom{10^9}{k}$ is less than the binomial coefficient $\binom{10^9+1}{k-1}$. Let a be the first (from the left) digit of k and let b be the second (from the left) digit of k. What is the value of $10a + b$?

15. Kate has two bags X and Y. Bag X contains 5 red marbles (and nothing else). Bag Y contains 4 red marbles and 1 blue marble (and nothing else). Kate chooses one of her bags at random (each with probability $\frac{1}{2}$) and removes a random marble from that bag (each marble in that bag being equally likely). She repeats the previous step until one of the bags becomes empty. At that point, what is the probability that the blue marble is still in bag Y? Express your answer as a fraction in simplest form.

16. Say that a complex number z is *three-presentable* if there is a complex number w of absolute value 3 such that $z = w - \frac{1}{w}$. Let T be the set of all three-presentable complex numbers. The set T forms a closed curve in the complex plane. What is the area inside T? Express your answer in the form $\frac{a}{b}\pi$, where a and b are positive, relatively-prime integers.

17. How many ordered triples (a, b, c), where a, b, and c are from the set $\{1, 2, 3, \ldots, 17\}$, satisfy the equation

$$a^3 + b^3 + c^3 + 2abc = a^2b + a^2c + b^2c + ab^2 + ac^2 + bc^2\,?$$

18. Sherry starts at the number 1. Whenever she's at 1, she moves one step up (to 2). Whenever she's at a number strictly between 1 and 10, she moves one step up or one step down, each with probability $\frac{1}{2}$. When she reaches 10, she stops. What is the expected number (average number) of steps that Sherry will take?

19. Define $L(x) = x - \frac{x^2}{2}$ for every real number x. If n is a positive integer, define a_n by

$$a_n = L\Bigl(L\Bigl(L\Bigl(\cdots L\Bigl(\frac{17}{n}\Bigr)\cdots\Bigr)\Bigr)\Bigr),$$

where there are n iterations of L. For example,

$$a_4 = L\Bigl(L\Bigl(L\Bigl(L\Bigl(\frac{17}{4}\Bigr)\Bigr)\Bigr)\Bigr).$$

As n approaches infinity, what value does na_n approach? Express your answer as a fraction in simplest form.

20. There are 6 distinct values of x strictly between 0 and $\frac{\pi}{2}$ that satisfy the equation

$$\tan(15x) = 15\tan(x).$$

Call these 6 values r_1, r_2, r_3, r_4, r_5, and r_6. What is the value of the sum

$$\frac{1}{\tan^2 r_1} + \frac{1}{\tan^2 r_2} + \frac{1}{\tan^2 r_3} + \frac{1}{\tan^2 r_4} + \frac{1}{\tan^2 r_5} + \frac{1}{\tan^2 r_6}\,?$$

Express your answer as a fraction in simplest form.