The Advantage Testing Foundation

Math Prize for Girls at MIT

2012 Solutions

Problem 1 In the morning, Esther biked from home to school at an average speed of x miles per hour. In the afternoon, having lent her bike to a friend, Esther walked back home along the same route at an average speed of 3 miles per hour. Her average speed for the round trip was 5 miles per hour. What is the value of x?

Answer: 15.

Solution: Let d be the number of miles between home and school. Going from home to school takes $\frac{d}{x}$ hours. Going from school to home takes $\frac{d}{3}$ hours. The entire round trip takes $\frac{2d}{5}$ hours. So we get the equation

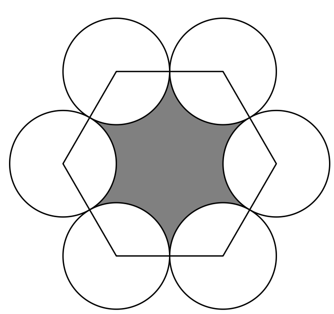
$$\frac{d}{x} + \frac{d}{3} = \frac{2d}{5} .$$

Dividing by d gives

$$\frac{1}{x} + \frac{1}{3} = \frac{2}{5} .$$

Solving for x gives $x = \boxed{15}$.

Problem 2 In the figure below, the centers of the six congruent circles form a regular hexagon with side length 2.



Adjacent circles are tangent to each other. What is the area of the shaded region? Express your answer in the form $a\sqrt{n}-b\pi$, where a and b are positive integers and n is a square-free positive integer.

Answer: $6\sqrt{3} - 2\pi$.

Solution: We can split the hexagon into 6 equilateral triangles with side length 2. Since the area of an equilateral triangle with side length x is $\frac{x^2\sqrt{3}}{4}$, the area of each of the 6 equilateral triangles is $\sqrt{3}$. So the area of the hexagon is $6\sqrt{3}$. The 6 circle sectors altogether form 2 full circles of radius 1. So the combined area of the sectors is 2π. Thus the area of the shaded region is $\boxed{6\sqrt{3} - 2\pi}$.

Problem 3 What is the least positive integer n such that $n!$ is a multiple of 2012^{2012}?

Answer: 1,010,527.

Solution: The prime factorization of 2012 is $2^2 \cdot 503$. So the prime factorization of 2012^{2012} is

$$(2^2 \cdot 503)^{2012} = 2^{4024} \cdot 503^{2012}.$$

Thus $n!$ is a multiple of 2012^{2012} if and only if it has at least 2012 factors of 503 and at least 4024 factors of 2.

If $n < 2009 \cdot 503$, then the number of factors of 503 in $n!$ is

$$\left\lfloor \frac{n}{503} \right\rfloor + \left\lfloor \frac{n}{503^2} \right\rfloor \leq 2008 + 3 = 2011 < 2012.$$

So we need $n \geq 2009 \cdot 503$. In that case, the number of factors of 503 in $n!$ is at least

$$\left\lfloor \frac{n}{503} \right\rfloor + \left\lfloor \frac{n}{503^2} \right\rfloor \geq 2009 + 3 = 2012.$$

Also, in that case, the number of factors of 2 in $n!$ is at least

$$\left\lfloor \frac{n}{2} \right\rfloor \geq \left\lfloor \frac{2009 \cdot 503}{2} \right\rfloor \geq 500{,}000 > 4024.$$

So the least value of n that works is

$$2009 \cdot 503 = (2000+9)(500+3) = 1{,}000{,}000 + 4500 + 6000 + 27 = \boxed{1{,}010{,}527}.$$

Problem 4 Evaluate the expression

$$\frac{121\left(\frac{1}{13}-\frac{1}{17}\right)+169\left(\frac{1}{17}-\frac{1}{11}\right)+289\left(\frac{1}{11}-\frac{1}{13}\right)}{11\left(\frac{1}{13}-\frac{1}{17}\right)+13\left(\frac{1}{17}-\frac{1}{11}\right)+17\left(\frac{1}{11}-\frac{1}{13}\right)}.$$

Answer: 41.

Solution: Let $a = 11$, let $b = 13$, and let $c = 17$. Using these variables, the expression becomes

$$\frac{a^2\left(\frac{1}{b}-\frac{1}{c}\right)+b^2\left(\frac{1}{c}-\frac{1}{a}\right)+c^2\left(\frac{1}{a}-\frac{1}{b}\right)}{a\left(\frac{1}{b}-\frac{1}{c}\right)+b\left(\frac{1}{c}-\frac{1}{a}\right)+c\left(\frac{1}{a}-\frac{1}{b}\right)}.$$

By grouping the terms in the numerator and denominator with the same reciprocal, we can rewrite this expression as

$$\frac{\frac{1}{a}(c^2-b^2)+\frac{1}{b}(a^2-c^2)+\frac{1}{c}(b^2-a^2)}{\frac{1}{a}(c-b)+\frac{1}{b}(a-c)+\frac{1}{c}(b-a)}.$$

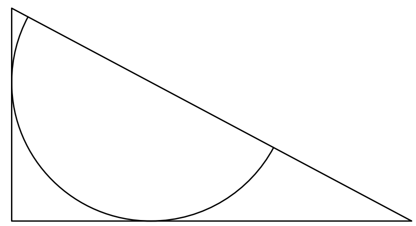
Using difference of squares, and letting $S = a + b + c$, the numerator of this fraction can be rewritten as follows:

$$\begin{aligned}
&\frac{1}{a}(c-b)(c+b)+\frac{1}{b}(a-c)(a+c)+\frac{1}{c}(b-a)(b+a)\\
&\quad=\frac{1}{a}(c-b)(S-a)+\frac{1}{b}(a-c)(S-b)+\frac{1}{c}(b-a)(S-c)\\
&\quad=\frac{1}{a}(c-b)S-(c-b)+\frac{1}{b}(a-c)S-(a-c)+\frac{1}{c}(b-a)S-(b-a)\\
&\quad=\left[\frac{1}{a}(c-b)+\frac{1}{b}(a-c)+\frac{1}{c}(b-a)\right]S.
\end{aligned}$$

But this final expression is the denominator of our fraction times S. So fraction simplifies to just S. The value of the fraction is

$$S = a + b + c = 11 + 13 + 17 = \boxed{41}.$$

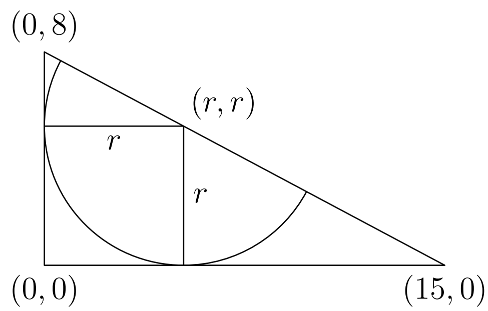
Problem 5 The figure below shows a semicircle inscribed in a right triangle.



The triangle has legs of length 8 and 15. The semicircle is tangent to the two legs, and its diameter is on the hypotenuse. What is the radius of the semicircle? Express your answer as a fraction in simplest form.

Answer: $\dfrac{120}{23}$.

Solution:



In the figure above, we have drawn radii to the two points where the semicircle is tangent to the legs. Let r be the radius of the semicircle.

We have imposed a coordinate system on the figure. The hypotenuse has endpoints $(15, 0)$ and $(0, 8)$. Using the intercept-intercept form, we see that the hypotenuse lies on the line $\frac{x}{15} + \frac{y}{8} = 1$.

The center of the semicircle is (r, r). Because the center is on the hypotenuse, we have $\frac{r}{15} + \frac{r}{8} = 1$. We can add the two fractions to get $\frac{23r}{120} = 1$. So $r = \boxed{\dfrac{120}{23}}$.

Problem 6 For how many ordered pairs of positive integers (x, y) is the least common multiple of x and y equal to 1,003,003,001?

Answer: 343.

Solution: Let's first factorize 1,003,003,001 into primes:

$$1{,}003{,}003{,}001 = 1001^3 = (7 \cdot 11 \cdot 13)^3 = 7^3 \cdot 11^3 \cdot 13^3.$$

For the least common multiple (lcm) of x and y to have this factorization, x must have the form $7^{a_1}11^{a_2}13^{a_3}$ and y must have the form $7^{b_1}11^{b_2}13^{b_3}$. We then have

$$7^3 \cdot 11^3 \cdot 13^3 = \operatorname{lcm}(x, y) = 7^{\max(a_1,b_1)}11^{\max(a_2,b_2)}13^{\max(a_3,b_3)}.$$

So $\max(a_1, b_1) = \max(a_2, b_2) = \max(a_3, b_3) = 3$.

To make $\max(a_1, b_1) = 3$, there are 7 choices for (a_1, b_1):

$$(0, 3), (1, 3), (2, 3), (3, 3), (3, 2), (3, 1), (3, 0).$$

Similarly, there are 7 choices for (a_2, b_2) and 7 choices for (a_3, b_3). Altogether, the number of choices is $7 \cdot 7 \cdot 7 = \boxed{343}$.

Problem 7 Let f_1, f_2, f_3, ..., be a sequence of numbers such that

$$f_n = f_{n-1} + f_{n-2}$$

for every integer $n \geq 3$. If $f_7 = 83$, what is the sum of the first 10 terms of the sequence?

Answer: 913.

Solution: Let's express each of the first 10 terms using only f_1 and f_2:

$$\begin{aligned}
f_1 &= f_1 \\
f_2 &= f_2 \\
f_3 &= f_2 + f_1 \\
f_4 &= 2f_2 + f_1 \\
f_5 &= 3f_2 + 2f_1 \\
f_6 &= 5f_2 + 3f_1 \\
f_7 &= 8f_2 + 5f_1 \\
f_8 &= 13f_2 + 8f_1 \\
f_9 &= 21f_2 + 13f_1 \\
f_{10} &= 34f_2 + 21f_1 \,.
\end{aligned}$$

(Did you notice that the coefficients are Fibonacci numbers?) Adding both sides, we see that the sum of the first 10 terms is equal to $88f_2 + 55f_1$. Therefore, the sum of the first 10 terms is

$$88f_2 + 55f_1 = 11(8f_2 + 5f_1) = 11f_7 = 11 \cdot 83 = \boxed{913}.$$

Problem 8 Suppose that x, y, and z are real numbers such that $x+y+z=3$ and $x^2+y^2+z^2=6$. What is the largest possible value of z? Express your answer in the form $a+\sqrt{b}$, where a and b are positive integers.

Answer: $1+\sqrt{2}$.

Solution: The two equations can be rewritten as $x+y=3-z$ and $x^2+y^2=6-z^2$. We can get an inequality for z as follows:

$$2(6-z^2)=2(x^2+y^2)=(x+y)^2+(x-y)^2 \geq (x+y)^2=(3-z)^2.$$

Expanding, we get

$$12-2z^2=2(6-z^2) \geq (3-z)^2=z^2-6z+9.$$

Grouping like terms gives

$$3z^2-6z \leq 3,$$

and so

$$z^2-2z \leq 1.$$

By completing the square, we get

$$(z-1)^2=z^2-2z+1 \leq 1+1=2.$$

Taking the square root of both sides gives

$$|z-1| \leq \sqrt{2}.$$

So $z \leq 1+\sqrt{2}$.

To see that $z=1+\sqrt{2}$ is possible, we can choose $x=y=\frac{3-z}{2}$. So the largest possible value of z is $\boxed{1+\sqrt{2}}$.

Problem 9 Bianca has a rectangle whose length and width are distinct primes less than 100. Let P be the perimeter of her rectangle, and let A be the area of her rectangle. What is the least possible value of $\frac{P^2}{A}$? Express your answer as a fraction in simplest form.

Answer: $\dfrac{82944}{5183}$.

Solution: Let ℓ be the length of the rectangle and w be the width. Without loss of generality, assume that $\ell > w$. The perimeter is $P = 2\ell + 2w$. The area is $A = \ell w$. So we have

$$\frac{P^2}{A} = \frac{(2\ell + 2w)^2}{\ell w} = \frac{4\ell^2 + 8\ell w + 4w^2}{\ell w} = 8 + 4\left(\frac{\ell}{w} + \frac{w}{\ell}\right).$$

To minimize this expression, we want $\frac{\ell}{w}$ to be as small as possible.

If ℓ and w are 1 apart, then $\ell = 3$ and $w = 2$. In that case, the ratio $\frac{\ell}{w}$ is $\frac{3}{2}$.

If ℓ and w are 2 apart, then they are twin primes. The largest pair of twin primes less than 100 is 73 and 71. So the ratio $\frac{\ell}{w}$ is at least $\frac{73}{71}$.

If ℓ and w are at least 3 apart, then

$$\frac{\ell}{w} \geq \frac{w+3}{w} = 1 + \frac{3}{w} > 1 + \frac{3}{100} > 1 + \frac{2}{71} = \frac{73}{71}.$$

Altogether, we have shown that $\frac{\ell}{w} \geq \frac{73}{71}$, with equality if $\ell = 73$ and $w = 71$. So we have

$$\frac{P^2}{A} = 8 + 4\left(\frac{\ell}{w} + \frac{w}{\ell}\right) \geq 8 + 4\left(\frac{73}{71} + \frac{71}{73}\right) = \frac{[2(73+71)]^2}{73 \cdot 71}.$$

The numerator is

$$[2(73+71)]^2 = (2 \cdot 144)^2 = (2^5 \cdot 3^2)^2 = 2^{10} \cdot 3^4 = 1024 \cdot 81 = 82{,}944.$$

The denominator is $73 \cdot 71 = 5183$. So we have

$$\frac{P^2}{A} \geq \frac{82{,}944}{5183},$$

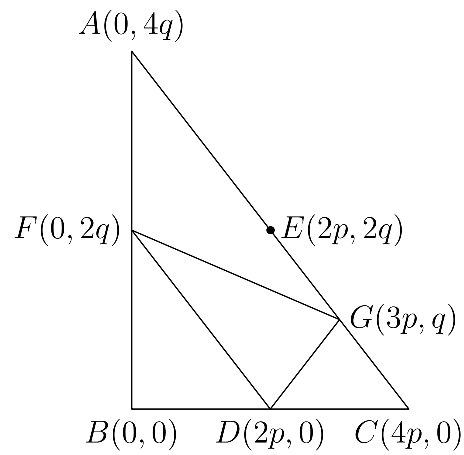
with equality if $\ell = 73$ and $w = 71$. Hence the least possible value of $\frac{P^2}{A}$ is $\boxed{\frac{82{,}944}{5183}}$.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 10 Let $\triangle ABC$ be a triangle with a right angle $\angle ABC$. Let D be the midpoint of $\overline{BC}$, let E be the midpoint of $\overline{AC}$, and let F be the midpoint of $\overline{AB}$. Let G be the midpoint of $\overline{EC}$. One of the angles of $\triangle DFG$ is a right angle. What is the least possible value of $\frac{BC}{AG}$? Express your answer as a fraction in simplest form.

Answer: $\dfrac{2}{3}$.

Solution:



In the figure above, we have imposed a coordinate system. We have placed point B at $(0, 0)$, point C at $(4p, 0)$, and point A at $(0, 4q)$, with p and q positive. (We inserted the factors of 4 to avoid fractions in the coordinates of the other points.) Because D is the midpoint of $\overline{BC}$, it is at $(2p, 0)$. Because E is the midpoint of $\overline{AC}$, it is at $(2p, 2q)$. Because F is the midpoint of $\overline{AB}$, it is at $(0, 2q)$. Because G is the midpoint of $\overline{EC}$, it is at $(3p, q)$.

The length of $\overline{BC}$ is $4p$. The length of $\overline{AG}$ is $3\sqrt{p^2 + q^2}$. The desired ratio is

$$\frac{BC}{AG} = \frac{4p}{3\sqrt{p^2 + q^2}} = \frac{4}{3\sqrt{1 + (q/p)^2}} \, .$$

This ratio is a decreasing function of $\frac{q}{p}$.

The slope of $\overline{DF}$ is $-\frac{q}{p}$, the slope of $\overline{FG}$ is $-\frac{q}{3p}$, and the slope of $\overline{DG}$ is $\frac{q}{p}$. Hence the slope of a line perpendicular to $\overline{DG}$ is $-\frac{p}{q}$, the negative reciprocal of $\frac{q}{p}$.

We are given that one of the angles of $\triangle DFG$ is a right angle. Because both $\overline{DF}$ and $\overline{FG}$ have negative slopes, they can't be perpendicular to each other. Hence either $\overline{DF}$ or $\overline{FG}$ is perpendicular to $\overline{DG}$. By our previous paragraph on slopes, we have either $-\frac{q}{p} = -\frac{p}{q}$ or $-\frac{q}{3p} = -\frac{p}{q}$. So $\frac{q}{p}$ is either 1 or $\sqrt{3}$. (All the steps above are reversible, so both choices are geometrically realizable.)

To make the desired ratio $\frac{BC}{AG}$ as small as possible, we should select the bigger choice, $\sqrt{3}$, for $\frac{q}{p}$. In that case,

$$\frac{BC}{AG} = \frac{4}{3\sqrt{1+(q/p)^2}} = \frac{4}{3\sqrt{1+3}} = \boxed{\frac{2}{3}}.$$

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 11 Alison has an analog clock whose hands have the following lengths: a inches (the hour hand), b inches (the minute hand), and c inches (the second hand), with $a < b < c$. The numbers a, b, and c are consecutive terms of an arithmetic sequence. The tips of the hands travel the following distances during a day: A inches (the hour hand), B inches (the minute hand), and C inches (the second hand). The numbers A, B, and C (in this order) are consecutive terms of a geometric sequence. What is the value of $\frac{B}{A}$? Express your answer in the form $x + y\sqrt{n}$, where x and y are positive integers and n is a square-free positive integer.

Answer: $60 + 24\sqrt{5}$.

Solution: In one day, the hour hand makes 2 revolutions, so $A = 2(2\pi a) = 4\pi a$. The minute hand makes 24 revolutions, so $B = 24(2\pi b) = 48\pi b$. The second hand makes $24 \cdot 60 = 1440$ revolutions, so $C = 1440(2\pi c) = 2880\pi c$.

Let r be the common ratio of the geometric sequence A, B, and C. Because $B = Ar$, we have $48\pi b = 4\pi ar$, which simplifies to $12b = ar$. Because $a < b$, we deduce that $r > 12$. Because $C = Ar^2$, we have $2880\pi c = (4\pi a)r^2$, which simplifies to $720c = ar^2$.

Because a, b, and c form an arithmetic sequence, we have $a + c = 2b$. By the previous paragraph, we have

$$a + \frac{1}{720}ar^2 = \frac{1}{6}ar,$$

which simplifies to $r^2 - 120r + 720 = 0$. By the quadratic formula, we have

$$r = \frac{120 \pm \sqrt{120^2 - 4(720)}}{2} = 60 \pm \sqrt{60^2 - 720} = 60 \pm 24\sqrt{5}\,.$$

Because $r > 12$, we have $r = \boxed{60 + 24\sqrt{5}}$.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 12 What is the sum of all positive integer values of n that satisfy the equation

$$\cos\left(\frac{\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right)\cos\left(\frac{4\pi}{n}\right)\cos\left(\frac{8\pi}{n}\right)\cos\left(\frac{16\pi}{n}\right) = \frac{1}{32}\,?$$

Answer: 47.

Solution: This problem involves the cosine of angles in a doubling sequence. That suggests using a double-angle identity such as

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

To use that identity, let's multiply both sides of the given equation by $\sin\frac{\pi}{n}$:

$$\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right)\cos\left(\frac{4\pi}{n}\right)\cos\left(\frac{8\pi}{n}\right)\cos\left(\frac{16\pi}{n}\right) = \frac{1}{32}\sin\left(\frac{\pi}{n}\right).$$

We can collapse the left-hand side by using the double-angle identity many times:

$$\begin{aligned}
&\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right)\cos\left(\frac{4\pi}{n}\right)\cos\left(\frac{8\pi}{n}\right)\cos\left(\frac{16\pi}{n}\right)\\
&\quad= \frac{1}{2}\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right)\cos\left(\frac{4\pi}{n}\right)\cos\left(\frac{8\pi}{n}\right)\cos\left(\frac{16\pi}{n}\right)\\
&\quad= \frac{1}{4}\sin\left(\frac{4\pi}{n}\right)\cos\left(\frac{4\pi}{n}\right)\cos\left(\frac{8\pi}{n}\right)\cos\left(\frac{16\pi}{n}\right)\\
&\quad= \frac{1}{8}\sin\left(\frac{8\pi}{n}\right)\cos\left(\frac{8\pi}{n}\right)\cos\left(\frac{16\pi}{n}\right)\\
&\quad= \frac{1}{16}\sin\left(\frac{16\pi}{n}\right)\cos\left(\frac{16\pi}{n}\right)\\
&\quad= \frac{1}{32}\sin\left(\frac{32\pi}{n}\right).
\end{aligned}$$

So our equation becomes

$$\frac{1}{32}\sin\left(\frac{32\pi}{n}\right) = \frac{1}{32}\sin\left(\frac{\pi}{n}\right).$$

We can remove the factors of $\frac{1}{32}$ on both sides:

$$\sin\left(\frac{32\pi}{n}\right) = \sin\left(\frac{\pi}{n}\right).$$

One way to proceed is to use the sum-to-product identity

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right).$$

Applying this identity with $\alpha = \frac{32\pi}{n}$ and $\beta = \frac{\pi}{n}$, we get the equation

$$2\sin\left(\frac{31\pi}{2n}\right)\cos\left(\frac{33\pi}{2n}\right) = 0.$$

So either $\sin\frac{31\pi}{2n}$ or $\cos\frac{33\pi}{2n}$ is 0.

If $\sin\frac{31\pi}{2n} = 0$, then $\frac{31\pi}{2n}$ is a multiple of π. So 31 is a multiple of $2n$. But 31 is odd, so this case has no solutions.

If $\cos\frac{33\pi}{2n} = 0$, then $\frac{33\pi}{2n}$ is $\frac{\pi}{2}$ plus a multiple of π. So 33 is an odd number times n. That means n is either 1, 3, 11, or 33.

Putting both cases together, we see that n is either 1, 3, 11, or 33. All the steps above are reversible, except for multiplying both sides of the original equation by $\sin\frac{\pi}{n}$. If $n > 1$, then this factor is positive, but if $n = 1$, then this factor is zero. Indeed, when we plug $n = 1$ into the original equation, we see that it is not a solution. Hence there are exactly three solutions to the original equation: 3, 11, and 33. Their sum is $3 + 11 + 33 = \boxed{47}$.

Problem 13 For how many integers n with $1 \le n \le 2012$ is the product

$$\prod_{k=0}^{n-1}\left(\left(1 + e^{2\pi ik/n}\right)^n + 1\right)$$

equal to zero?

Answer: 335.

Solution: If the product is zero, then one of the factors $(1 + e^{2\pi ik/n})^n + 1$ is zero. In that case, $(1 + e^{2\pi ik/n})^n$ is -1. That implies $1 + e^{2\pi ik/n}$ has absolute value 1. So the three complex numbers -1, 0, and $e^{2\pi ik/n}$ are the vertices of an equilateral triangle with side length 1. Thus $e^{2\pi ik/n}$ is either $e^{2\pi i/3}$ or its conjugate.

In that case, $1 + e^{2\pi ik/n}$ is either $e^{\pi i/3}$ or its conjugate. So $(1 + e^{2\pi ik/n})^n$ is either $e^{n\pi i/3}$ or its conjugate. The only way that can be -1 is if n is an odd multiple of 3. (And if n is an odd multiple of 3, then the factor corresponding to $k = n/3$ will be zero.)

So our problem boils down to counting the odd multiples of 3 between 1 and 2012. There are 670 multiples of 3 in this interval. Thus the number of odd multiples of 3 in this interval is half that number, or $\boxed{335}$.

Problem 14 Let k be the smallest positive integer such that the binomial coefficient $\binom{10^9}{k}$ is less than the binomial coefficient $\binom{10^9+1}{k-1}$. Let a be the first (from the left) digit of k and let b be the second (from the left) digit of k. What is the value of $10a+b$?

Answer: 38.

Solution: Let $n = 10^9 + 1$. Our inequality is

$$\binom{n-1}{k} < \binom{n}{k-1}.$$

We will use the following two identities on binomial coefficients:

$$\begin{aligned} n\binom{n-1}{k} &= (n-k)\binom{n}{k} \\ (n-k+1)\binom{n}{k-1} &= k\binom{n}{k}. \end{aligned}$$

Combining these two identities gives a third identity

$$nk\binom{n-1}{k} = (n-k+1)(n-k)\binom{n}{k-1}.$$

So our desired inequality is equivalent to

$$(n-k+1)(n-k) < nk.$$

To get a feeling for this inequality, let's temporarily ignore the "+1" and change the inequality to an equation:

$$(n-k)^2 = nk.$$

This equation is quadratic in k, and so can be solved by the quadratic formula. The smaller root is $k = \alpha n$, where $\alpha = \frac{3-\sqrt{5}}{2}$ is the smaller root of $(1-\alpha)^2 = \alpha$. Because $\sqrt{5}$ is between 2.23 and 2.24, the value of α is between 0.38 and 0.385. So we expect the smallest integer k that satisfies the inequality starts with 38. We will verify this claim rigorously below.

First, we will show that if $k \le \alpha n$, then the inequality fails:

$$(n-k+1)(n-k) \ge (n-k)^2 \ge (1-\alpha)^2 n^2 = \alpha n^2 \ge nk.$$

Second, we will show that if $k \geq \alpha(n+1)$, then the inequality holds:

$$(n-k+1)(n-k) < (1-\alpha)(n+1)(1-\alpha)n = \alpha(n+1)n \leq kn.$$

So the smallest positive integer k that satisfies the inequality is between αn and $\alpha(n+1)+1$. Hence a (the first digit of k) is indeed 3, and b (the second digit of k) is 8. Therefore $10a+b$ is $\boxed{38}$.

Problem 15 Kate has two bags X and Y. Bag X contains 5 red marbles (and nothing else). Bag Y contains 4 red marbles and 1 blue marble (and nothing else). Kate chooses one of her bags at random (each with probability $\frac{1}{2}$) and removes a random marble from that bag (each marble in that bag being equally likely). She repeats the previous step until one of the bags becomes empty. At that point, what is the probability that the blue marble is still in bag Y? Express your answer as a fraction in simplest form.

Answer: $\dfrac{63}{256}$.

Solution: Let k be an integer between 1 and 5 (inclusive). We will first compute the probability that at the end there are exactly k marbles in bag Y. The final marble removed must have been from bag X. Before that, 4 marbles were removed from bag X and $5-k$ marbles were removed from bag Y. There are $\binom{9-k}{4}$ ways of choosing the bag sequence. The probability of making a particular choice is $(1/2)^{10-k}$. So the probability that at the end there are exactly k marbles in bag Y is

$$\frac{\binom{9-k}{4}2^k}{2^{10}}.$$

We can use this result to find the expected value of the number of marbles remaining in bag Y. The expected value is

$$\frac{1}{2^{10}}\sum_{k=1}^{5}\binom{9-k}{4}2^k k.$$

Let's compute the sum:

$$\begin{aligned}\sum_{k=1}^{5}\binom{9-k}{4}2^k k &= 2\binom{8}{4} + 8\binom{7}{4} + 24\binom{6}{4} + 64\binom{5}{4} + 160\binom{4}{4} \\ &= 2\cdot 70 + 8\cdot 35 + 24\cdot 15 + 64\cdot 5 + 160\cdot 1 \\ &= 140 + 280 + 360 + 320 + 160 \\ &= 1260.\end{aligned}$$

Hence the expected number of marbles remaining in bag Y is

$$\frac{1260}{2^{10}}.$$

The blue marble is 1 of the 5 marbles in bag Y. So the probability that the blue marble remains in bag Y is $\frac{1}{5}$ of the expected number of marbles remaining in bag Y:

$$\frac{1260}{5 \cdot 2^{10}} = \frac{252}{2^{10}} = \frac{63}{2^8} = \boxed{\frac{63}{256}}.$$

Note: The answer happens to equal $\binom{10}{5}/2^{10}$. That's not an accident! See `www.aops.com/Forum/viewtopic.php?t=499689` for an explanation.

Problem 16 Say that a complex number z is *three-presentable* if there is a complex number w of absolute value 3 such that $z = w - \frac{1}{w}$. Let T be the set of all three-presentable complex numbers. The set T forms a closed curve in the complex plane. What is the area inside T? Express your answer in the form $\frac{a}{b}\pi$, where a and b are positive, relatively-prime integers.

Answer: $\dfrac{80}{9}\pi$.

Solution: Let w be a complex number with absolute value 3. Let z be the three-presentable number $w - \frac{1}{w}$. Then z can be rewritten as

$$z = w - \frac{1}{w} = w - \frac{\overline{w}}{|w|^2} = w - \frac{\overline{w}}{9}.$$

Let $w = x + yi$, where x and y are real numbers. By the previous equation, we have

$$z = x + yi - \frac{x - yi}{9} = \frac{8}{9}x + \frac{10}{9}yi.$$

In other words, to go from w to z, stretch the real part by a factor of $\frac{8}{9}$ and stretch the imaginary part by a factor of $\frac{10}{9}$. So T (the set of three-presentable numbers) is an ellipse formed by stretching a circle of radius 3 by a factor of $\frac{8}{9}$ in the x-direction and by a factor of $\frac{10}{9}$ in the y-direction. The area inside a circle of radius 3 is 9π. So the area inside T is

$$\frac{8}{9} \cdot \frac{10}{9} \cdot 9\pi = \boxed{\frac{80}{9}\pi}.$$

Problem 17 How many ordered triples (a, b, c), where a, b, and c are from the set $\{1, 2, 3, \ldots, 17\}$, satisfy the equation

$$a^3 + b^3 + c^3 + 2abc = a^2b + a^2c + b^2c + ab^2 + ac^2 + bc^2\,?$$

Answer: 408.

Solution: Let's move all the terms of the equation to the left side:

$$a^3 + b^3 + c^3 + 2abc - a^2b - a^2c - b^2c - ab^2 - ac^2 - bc^2 = 0.$$

We will try to factor the left side. To do so, let's view the left side as a cubic polynomial in a. We can group the terms as follows:

$$a^3 - (b + c)a^2 - (b^2 - 2bc + c^2)a + b^3 + c^3 - b^2c - bc^2 = 0.$$

We can factor the linear coefficient and the constant coefficient:

$$a^3 - (b + c)a^2 - (b - c)^2a + (b - c)^2(b + c) = 0.$$

That helps us partially factor the left side:

$$(a - b - c)\left[(b - c)^2 - a^2\right] = 0.$$

By difference of squares, we can fully factor:

$$(a - b - c)(b - a - c)(c - a - b) = 0.$$

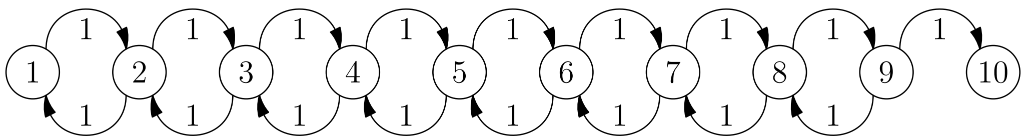
So either $a = b + c$ or $b = a + c$ or $c = a + b$. Because the variables are positive, those three cases are mutually exclusive. Hence we can count the number of solutions to $a = b + c$ and then multiply by 3.

Let's count the solutions to $a = b + c$, where each of the variables is an integer from 1 to 17. We have $\binom{17}{2}$ choices for a and b. For each such choice, we have exactly 1 choice for c. So the number of solutions to $a = b + c$ is $\binom{17}{2}$, or 136. As we said in the previous paragraph, we need to multiply this count by 3 to get the final answer. Hence the answer is $3 \cdot 136$, or $\boxed{408}$.

Problem 18 Sherry starts at the number 1. Whenever she's at 1, she moves one step up (to 2). Whenever she's at a number strictly between 1 and 10, she moves one step up or one step down, each with probability $\frac{1}{2}$. When she reaches 10, she stops. What is the expected number (average number) of steps that Sherry will take?

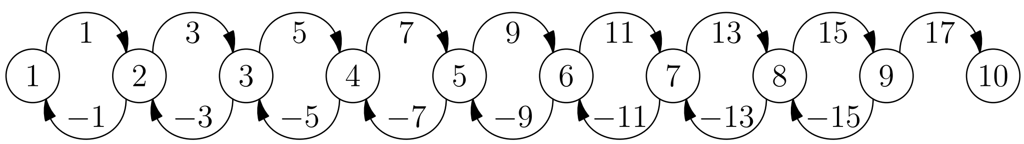
Answer: 81.

Solution: We can draw a state diagram to represent Sherry's trip:



Whichever state Sherry is in the diagram, she chooses an out-going edge at random as her next step. The "1" labels represent the cost of making a step. Since each cost label is 1, at each state the expected cost of one random step is 1.

Because the problem asks only about the expected cost of the entire trip, we can renumber the labels as long as the expected cost of one random step remains 1. Consider the following renumbering:



At each state, the expected cost of one random step is still 1. For example, at state 4, the out-going edges are labeled -5 and 7, for an average of 1. So we can analyze the expected cost of the trip under the new numbering.

In any path from 1 to 10 under the new numbering, there will be cancellation. For example, there might be several uses of the -5 and 5 labels, but there will always be exactly one more use of the 5 label than the -5 label. So the total cost will always be

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17.$$

The sum of these 9 odd numbers is 9^2, or 81. Because the total cost is always 81, the expected cost is $\boxed{81}$.

Problem 19 Define $L(x) = x - \frac{x^2}{2}$ for every real number x. If n is a positive integer, define a_n by

$$a_n = L\Bigl(L\Bigl(L\Bigl(\cdots L\Bigl(\frac{17}{n}\Bigr)\cdots\Bigr)\Bigr)\Bigr),$$

where there are n iterations of L. For example,

$$a_4 = L\Bigl(L\Bigl(L\Bigl(L\Bigl(\frac{17}{4}\Bigr)\Bigr)\Bigr)\Bigr).$$

As n approaches infinity, what value does na_n approach? Express your answer as a fraction in simplest form.

Answer: $\dfrac{34}{19}$.

Solution: From the definition of L, we see that L obeys the following three properties:

1. If $x \neq 0$ and $x \neq 2$, then
$$\frac{1}{L(x)} - \frac{1}{x} = \frac{1}{2-x}\,.$$
2. $L(x) \leq x$.
3. If $0 < x < 2$, then $L(x) > 0$.

Since we're interested in n approaching infinity, we may assume that $n \geq 9$, so $\frac{17}{n}$ is strictly between 0 and 2. If k is a nonnegative integer, let $L^{(k)}$ be the kth iterate of L. By Property 2, we have $L^{(k)}(\frac{17}{n}) \leq \frac{17}{n}$ for every k. By Property 3, we have $L^{(k)}(\frac{17}{n}) > 0$ for every k. Combining these last two inequalities, we have
$$\frac{1}{2} < \frac{1}{2 - L^{(k)}(17/n)} \leq \frac{1}{2 - 17/n}\,.$$
Hence by Property 1, we have
$$\frac{1}{2} < \frac{1}{L^{(k+1)}(17/n)} - \frac{1}{L^{(k)}(17/n)} \leq \frac{1}{2 - 17/n}\,.$$

By summing the previous inequality for all k from 0 to $n-1$, we have
$$\frac{n}{2} < \frac{1}{L^{(n)}(17/n)} - \frac{1}{17/n} \leq \frac{n}{2 - 17/n}\,.$$
Because a_n is $L^{(n)}(17/n)$, we have
$$\frac{n}{2} < \frac{1}{a_n} - \frac{n}{17} \leq \frac{n}{2 - 17/n}\,.$$
Dividing by n gives
$$\frac{1}{2} < \frac{1}{na_n} - \frac{1}{17} \leq \frac{1}{2 - 17/n}\,.$$

As n approaches infinity, the left and right expressions approach $\frac{1}{2}$. Hence the middle expression does too. So $\frac{1}{na_n}$ approaches $\frac{1}{2}+\frac{1}{17}$, which is $\frac{19}{34}$. Therefore, na_n approaches $\boxed{\dfrac{34}{19}}$.

Problem 20 There are 6 distinct values of x strictly between 0 and $\frac{\pi}{2}$ that satisfy the equation

$$\tan(15x) = 15\tan(x).$$

Call these 6 values r_1, r_2, r_3, r_4, r_5, and r_6. What is the value of the sum

$$\frac{1}{\tan^2 r_1} + \frac{1}{\tan^2 r_2} + \frac{1}{\tan^2 r_3} + \frac{1}{\tan^2 r_4} + \frac{1}{\tan^2 r_5} + \frac{1}{\tan^2 r_6}\,?$$

Express your answer as a fraction in simplest form.

Answer: $\dfrac{78}{5}$.

Solution: We will study the more general equation $\tan(nx) = n\tan(x)$, where n is a fixed integer greater than 1. The multiple-angle formula for tangent says that

$$\tan nx = \frac{\sum_{k=0}^{\infty}(-1)^k\binom{n}{2k+1}(\tan x)^{2k+1}}{\sum_{k=0}^{\infty}(-1)^k\binom{n}{2k}(\tan x)^{2k}}\,.$$

If $\tan(nx) = n\tan(x)$, then the formula becomes

$$n\tan x = \frac{\sum_{k=0}^{\infty}(-1)^k\binom{n}{2k+1}(\tan x)^{2k+1}}{\sum_{k=0}^{\infty}(-1)^k\binom{n}{2k}(\tan x)^{2k}}\,.$$

Clearing the fraction, we get

$$n\sum_{k=0}^{\infty}(-1)^k\binom{n}{2k}(\tan x)^{2k+1} = \sum_{k=0}^{\infty}(-1)^k\binom{n}{2k+1}(\tan x)^{2k+1}\,.$$

Bringing all the terms to one side, we get

$$\sum_{k=0}^{\infty}(-1)^k\left[n\binom{n}{2k} - \binom{n}{2k+1}\right](\tan x)^{2k+1} = 0.$$

We can simplify the bracketed expression using the binomial identity

$$n\binom{n}{2k} - \binom{n}{2k+1} = 2k\binom{n+1}{2k+1}.$$

With this identity, our main equation becomes

$$\sum_{k=0}^{\infty}(-1)^k 2k\binom{n+1}{2k+1}(\tan x)^{2k+1} = 0.$$

The $k = 0$ term is 0 and so we can remove it:

$$\sum_{k=1}^{\infty}(-1)^k 2k\binom{n+1}{2k+1}(\tan x)^{2k+1} = 0.$$

Because x is acute, $\tan x$ is positive, and so we can divide both sides by $2(\tan x)^3$:

$$\sum_{k=1}^{\infty}(-1)^k k\binom{n+1}{2k+1}(\tan x)^{2(k-1)} = 0.$$

Let $t = \tan^2 x$. The equation becomes

$$\sum_{k=1}^{\infty}(-1)^k k\binom{n+1}{2k+1}t^{k-1} = 0.$$

The constant coefficient of the left-hand side is $-\binom{n+1}{3}$. The linear coefficient is $2\binom{n+1}{5}$. So by Vieta's formula, the sum of the reciprocals of the roots of t is

$$\frac{2\binom{n+1}{5}}{\binom{n+1}{3}} = \frac{(n-2)(n-3)}{10}.$$

When $n = 15$, this sum is

$$\frac{13 \cdot 12}{10} = \frac{13 \cdot 6}{5} = \boxed{\frac{78}{5}}.$$

Note: By letting n approach infinity, we can show that the squared reciprocals of the positive solutions to $\tan x = x$ add up to $\frac{1}{10}$.