The Advantage Testing Foundation
Math Prize for Girls Olympiad

Thursday, November 14, 2013

Time Limit: 4 hours

1. Let n be a positive integer. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $-1 \le a_i \le 1$ (for all $1 \le i \le n$). Let $b_1, b_2, \ldots, b_n$ be real numbers such that $-1 \le b_i \le 1$ (for all $1 \le i \le n$). Prove that

$$\left| \prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i \right| \le \sum_{i=1}^{n} |a_i - b_i| \ .$$

2. Say that a (nondegenerate) triangle is *funny* if it satisfies the following condition: the altitude, median, and angle bisector drawn from one of the vertices divide the triangle into 4 non-overlapping triangles whose areas form (in some order) a 4-term arithmetic sequence. (One of these 4 triangles is allowed to be degenerate.) Find with proof all funny triangles.

3. 10000 nonzero digits are written in a 100-by-100 table, one digit per cell. From left to right, each row forms a 100-digit integer. From top to bottom, each column forms a 100-digit integer. So the rows and columns form 200 integers (each with 100 digits), not necessarily distinct. Prove that if at least 199 of these 200 numbers are divisible by 2013, then all of them are divisible by 2013.

4. We are given a finite set of segments of the same line. Prove that we can color each segment red or blue such that, for each point p on the line, the number of red segments containing p differs from the number of blue segments containing p by at most 1.