The Advantage Testing Foundation

Math Prize for Girls at MIT

2013 Olympiad Solutions

Problem 1 Let n be a positive integer. Let a_1, a_2, ..., a_n be real numbers such that $-1 \le a_i \le 1$ (for all $1 \le i \le n$). Let b_1, b_2, ..., b_n be real numbers such that $-1 \le b_i \le 1$ (for all $1 \le i \le n$). Prove that

$$\left| \prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i \right| \le \sum_{i=1}^{n} |a_i - b_i| \; .$$

Solution: If k is an integer such that $0 \le k \le n$, define P_k to be the hybrid product

$$P_k = \prod_{i=1}^{k} a_i \times \prod_{j=k+1}^{n} b_j \, .$$

Note that $P_0 = \prod_{i=1}^{n} b_i$ and $P_n = \prod_{i=1}^{n} a_i$.

Let k be an integer such that $1 \le k \le n$. We will show that P_k and P_{k-1} are near each other. We can rewrite P_k as follows:

$$P_k = \prod_{i=1}^{k} a_i \times \prod_{j=k+1}^{n} b_j = a_k \times \prod_{i=1}^{k-1} a_i \times \prod_{j=k+1}^{n} b_j \, .$$

Similarly, we have

$$P_{k-1} = \prod_{i=1}^{k-1} a_i \times \prod_{j=k}^{n} b_j = b_k \times \prod_{i=1}^{k-1} a_i \times \prod_{j=k+1}^{n} b_j \, .$$

Subtracting the two equations gives

$$P_k - P_{k-1} = (a_k - b_k) \times \prod_{i=1}^{k-1} a_i \times \prod_{j=k+1}^{n} b_j \, .$$

Taking absolute value of both sides gives

$$|P_k - P_{k-1}| = |a_k - b_k| \times \prod_{i=1}^{k-1} |a_i| \times \prod_{j=k+1}^{n} |b_j| .$$

Because $|a_i| \le 1$ and $|b_j| \le 1$, we have

$$|P_k - P_{k-1}| \le |a_k - b_k| .$$

Now we are nearly done. The difference in the products of the a's and the b's can be expressed as a telescoping sum:

$$\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i = P_n - P_0 = \sum_{k=1}^{n} (P_k - P_{k-1}).$$

Taking the absolute value of both sides and using the triangle inequality, we have

$$\left| \prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i \right| = \left| \sum_{k=1}^{n} (P_k - P_{k-1}) \right| \le \sum_{k=1}^{n} |P_k - P_{k-1}| .$$

Because $|P_k - P_{k-1}| \le |a_k - b_k|$, we have

$$\left| \prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i \right| \le \sum_{k=1}^{n} |a_k - b_k| .$$

That's what we wanted to prove.

Problem 2 Say that a (nondegenerate) triangle is *funny* if it satisfies the following condition: the altitude, median, and angle bisector drawn from one of the vertices divide the triangle into 4 non-overlapping triangles whose areas form (in some order) a 4-term arithmetic sequence. (One of these 4 triangles is allowed to be degenerate.) Find with proof all funny triangles.

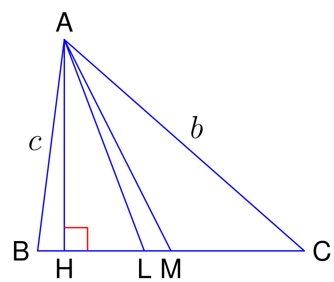
Solution: Let ABC be a funny triangle. Suppose that $\overline{AH}$, $\overline{AL}$, and $\overline{AM}$ are the given altitude, angle bisector, and median. Following the usual notation, let $b = AC$ and $c = AB$.

First, we claim that $b \neq c$. If $b = c$, then H, L, and M would be the same point, which means that the areas of the 4 sub-triangles would not form an arithmetic sequence. So $b \neq c$. Without loss of generality, assume that $c < b$.

Next, we figure out the relative order of the points H, L, and M. Because $c < b$, it follows that $\angle C < \angle B$. Hence $\angle CAH > \angle BAH$. So H is strictly closer to B than L is.

By the Angle Bisector theorem, $\frac{BL}{LC} = \frac{c}{b}$, which is less than 1. So L is strictly closer to B than M is.

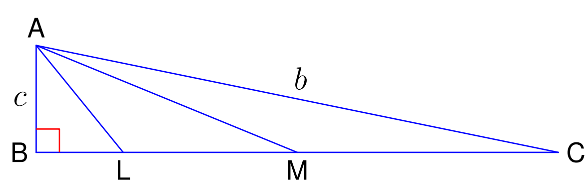
Hence, going from B to C, the points H, L, and M appear in that order. Here is a diagram of what we've learned.



The 4 sub-triangles share the same altitude $\overline{AH}$ and their areas form an arithmetic sequence (in some order). Hence their bases BH, HL, LM, and MC form an arithmetic sequence (in some order). Let the arithmetic sequence (in order) be x, $x + d$, $x + 2d$, and $x + 3d$, where $d \geq 0$. Because MC is the longest of the 4 bases, $MC = x + 3d$. Because M is the midpoint, we get the equation

$$x + (x + d) + (x + 2d) = x + 3d.$$

Simplifying gives $x = 0$. So one of the bases is degenerate. From our previous analysis, we see that HL, LM, and MC are all positive. So $BH = 0$. In other words, B and H are the same point, which means that $\angle ABC$ is a right angle. Below is a diagram with this new information.



Because $x = 0$, the arithmetic sequence becomes 0, d, $2d$, and $3d$. The longest base MC is $3d$. So BL and LM equal d and $2d$ in some order. We handle the two cases separately.

Consider the case $BL = d$ and $LM = 2d$. By the Angle Bisector theorem, we have

$$\frac{b}{c} = \frac{LC}{BL} = \frac{5d}{d} = 5.$$

So $b = 5c$. By the Pythagorean theorem, $a = c\sqrt{5^2 - 1} = 2\sqrt{6}c$. So the ratio of the sides is $1 : 2\sqrt{6} : 5$. Conversely, every triangle with that side ratio is funny.

Consider the case $BL = 2d$ and $LM = d$. By the Angle Bisector theorem, we have

$$\frac{b}{c} = \frac{LC}{BL} = \frac{4d}{2d} = 2.$$

So $b = 2c$. By the Pythagorean theorem, $a = c\sqrt{2^2 - 1} = \sqrt{3}c$. So the ratio of the sides is $1 : \sqrt{3} : 2$. (In other words, we have a 30-60-90 triangle.) Conversely, every triangle with that side ratio is funny.

Combining the two cases, we see that the funny triangles are those right triangles whose side lengths are in the ratio $1 : 2\sqrt{6} : 5$ or $1 : \sqrt{3} : 2$.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 3 10000 nonzero digits are written in a 100-by-100 table, one digit per cell. From left to right, each row forms a 100-digit integer. From top to bottom, each column forms a 100-digit integer. So the rows and columns form 200 integers (each with 100 digits), not necessarily distinct. Prove that if at least 199 of these 200 numbers are divisible by 2013, then all of them are divisible by 2013.

Solution: Let A be the given table. We will index the rows from 0 to 99, starting from the bottom. We will index the columns from 0 to 99, starting from the right. Let a_{ij} be the digit in row i and column j.

We will now look at the integers formed by the rows and columns. Let R_i be the 100-digit integer formed by row i. We have the base-10 formula

$$R_i = \sum_{j=0}^{99} 10^j a_{ij}\,.$$

Similarly, let C_j be the integer formed by column j. We have the formula

$$C_j = \sum_{i=0}^{99} 10^i a_{ij}\,.$$

We will find it convenient to look at a scaled version of table A. Let B be the 100-by-100 table with entries

$$b_{ij} = 10^{i+j} a_{ij} \,.$$

The sum of the entries in row i of B is

$$\sum_{j=0}^{99} b_{ij} = \sum_{j=0}^{99} 10^{i+j} a_{ij} = 10^i \sum_{j=0}^{99} 10^j a_{ij} = 10^i R_i \,.$$

Similarly, the sum of the entries in column j of B is

$$\sum_{i=0}^{99} b_{ij} = \sum_{i=0}^{99} 10^{i+j} a_{ij} = 10^j \sum_{i=0}^{99} 10^i a_{ij} = 10^j C_j \,.$$

The sum of the row sums of a table is equal to the sum of the column sums of the table. Hence we get the interesting equation

$$\sum_{i=0}^{99} 10^i R_i = \sum_{j=0}^{99} 10^j C_j \,.$$

Finally let's look at divisibility. We are given that at least 199 of the R_i and C_j are divisible by 2013. From the equation above, it follows that the remaining R_i or C_j, multiplied by a power of 10, is also divisible by 2013. Because 10 is relatively prime to 2013, the remaining R_i or C_j itself is divisible by 2013. That's what we wanted to prove.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 4 We are given a finite set of segments of the same line. Prove that we can color each segment red or blue such that, for each point p on the line, the number of red segments containing p differs from the number of blue segments containing p by at most 1.

Solution: We will view the given line as the real number line. So each given segment is a closed interval of real numbers.

Instead of red and blue, we will use the colors $+1$ and -1. So for each real number p, we want the sum of the colors of the intervals containing p to have absolute value at most 1.

Suppose the given intervals are $[a_1, b_1]$, $[a_2, b_2]$, …, $[a_n, b_n]$. We will define a graph on the points a_1, b_1, a_2, b_2, …, a_n, b_n. First, draw an edge between a_i and b_i (for all i from 1 to n); call these edges the "interval" edges. Second, list the points a_1, b_1, a_2, b_2, …, a_n, b_n from smallest to largest. (In case of ties, list a's before b's; otherwise break ties arbitrarily.) Draw an edge between the first and second points, the third and fourth points, and so on; call these edges the "consecutive" edges. The interval edges and consecutive edges together form our graph.

We claim that our graph is 2-colorable. Every point in the graph is incident to exactly one interval edge and exactly one consecutive edge. So every simple path in the graph alternates between interval edges and consecutive edges. Hence the graph consists of the disjoint union of even cycles. Therefore, the graph is 2-colorable, which means that each point can colored $+1$ or -1 such that adjacent points receive different colors. Fix such a coloring of the points. Let x_i be the color of a_i; let y_i be the color of b_i. Because a_i and b_i form an interval edge, x_i and y_i are negations of each other. Similarly, two points that form a consecutive edge have opposite colors.

As requested, we will now color the intervals. Define the color of interval $[a_i, b_i]$ to be x_i, the color of its left endpoint.

We claim that this coloring of intervals has the desired property. Let p be a real number. We need to show that the sum

$$\sum_{\substack{i \\ a_i \le p \le b_i}} x_i$$

has absolute value at most 1. We can rewrite this sum as

$$\sum_{\substack{i \\ a_i \le p}} x_i - \sum_{\substack{i \\ b_i < p}} x_i \, .$$

Because x_i and y_i are negations of each other, we get the equivalent expression

$$\sum_{\substack{i \\ a_i \le p}} x_i + \sum_{\substack{i \\ b_i < p}} y_i \, .$$

That expression is the sum of the colors of the points less than p (plus possibly the a's equal to p). Because two points that form a consecutive edge have opposite colors, this sum cancels out except possibly for the largest point. So the sum has absolute value at most 1. That's what we wanted to prove.