ADVANTAGE TESTING FOUNDATION

THE FIFTH ANNUAL MATH PRIZE FOR GIRLS

Saturday, September 7, 2013

TEST BOOKLET

Test Version A

DIRECTIONS

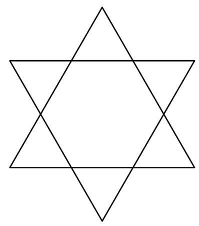
1. Do not open this test until your proctor instructs you to.
2. Fill out the top of your answer sheet. Your student number is on your name tag. Your test version (A, B, C, or D) is above on this page.
3. You may use pencils, pens, and erasers. You may use any available space on this test booklet for scratch work. You may NOT use calculators, rulers, protractors, compasses, graph paper, books, or slide rules.
4. Write your final answers on the answer sheet. When a problem specifies a particular form for the answer, write your answer in that form. Write your answers clearly. We will collect only the answer sheet from you.

NOTES

1. This test contains 20 problems. You will have 150 minutes (2.5 hours) to take the test. Your score will be the number of correct answers.
2. Figures are not necessarily drawn to scale.
3. Good luck!

Math Prize for Girls 2013

1. The figure below shows two equilateral triangles each with area 1.



The intersection of the two triangles is a regular hexagon. What is the area of the union of the two triangles? Express your answer as a fraction in simplest form.

2. When the binomial coefficient $\binom{125}{64}$ is written out in base 10, how many zeros are at the rightmost end?

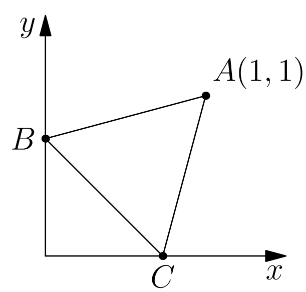
3. Let S_1, S_2, $\ldots$, S_{125} be 125 sets of 5 numbers each, comprising 625 distinct numbers. Let m_i be the median of S_i. Let M be the median of m_1, m_2, $\ldots$, m_{125}. What is the greatest possible number of the 625 numbers that are less than M?

4. The MathMatters competition consists of 10 players P_1, P_2, $\ldots$, P_{10} competing in a ladder-style tournament. Player P_{10} plays a game with P_9: the loser is ranked 10th, while the winner plays P_8. The loser of that game is ranked 9th, while the winner plays P_7. They keep repeating this process until someone plays P_1: the loser of that final game is ranked 2nd, while the winner is ranked 1st. How many different rankings of the players are possible?

5. Say that a 4-digit positive integer is *mixed* if it has 4 distinct digits, its leftmost digit is neither the biggest nor the smallest of the 4 digits, and its rightmost digit is not the smallest of the 4 digits. For example, 2013 is mixed. How many 4-digit positive integers are mixed?

6. Three distinct real numbers form (in some order) a 3-term arithmetic sequence, and also form (in possibly a different order) a 3-term geometric sequence. Compute the greatest possible value of the common ratio of this geometric sequence. Express your answer as a fraction in simplest form.

7. In the figure below, $\triangle ABC$ is an equilateral triangle.



Point A has coordinates $(1, 1)$, point B is on the positive y-axis, and point C is on the positive x-axis. What is the area of $\triangle ABC$? Express your answer in the form $a\sqrt{n} - b$, where a and b are positive integers and n is a square-free positive integer.

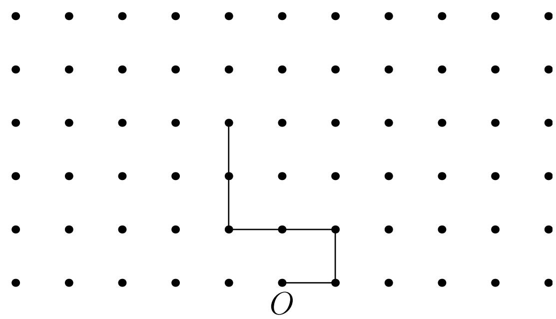
8. Let R be the set of points (x, y) such that x and y are positive, $x + y$ is at most 2013, and

$$\lceil x \rceil \lfloor y \rfloor = \lfloor x \rfloor \lceil y \rceil .$$

Compute the area of set R. Express your answer as a fraction in simplest form. Recall that $\lfloor a \rfloor$ is the greatest integer that is less than or equal to a, and $\lceil a \rceil$ is the least integer that is greater than or equal to a.

9. Let A and B be distinct positive integers such that each has the same number of positive divisors that 2013 has. Compute the least possible value of $|A-B|$.

10. The following figure shows a *walk* of length 6:



This walk has three interesting properties:

(a) It starts at the origin, labelled O.

(b) Each step is 1 unit north, east, or west. There are no south steps.

(c) The walk never comes back to a point it has been to.

Let's call a walk with these three properties a *northern walk*. There are 3 northern walks of length 1 and 7 northern walks of length 2. How many northern walks of length 6 are there?

11. Alice throws two standard dice, with A being the number on her first die and B being the number on her second die. She then draws the line $Ax + By = 2013$. Boris also throws two standard dice, with C being the number on his first die and D being the number on his second die. He then draws the line $Cx + Dy = 2014$. Compute the probability that these two lines are parallel. Express your answer as a fraction in simplest form.

12. The rectangular parallelepiped (box) P has some special properties. If one dimension of P were doubled and another dimension were halved, then the surface area of P would stay the same. If instead one dimension of P were tripled and another dimension were divided by 3, then the surface area of P would still stay the same. If the middle (by length) dimension of P is 1, compute the least possible volume of P. Express your answer as a fraction in simplest form.

13. Each of n boys and n girls chooses a random number from the set $\{1, 2, 3, 4, 5\}$, uniformly and independently. Let p_n be the probability that every boy chooses a different number than every girl. As n approaches infinity, what value does $\sqrt[n]{p_n}$ approach? Express your answer as a fraction in simplest form.

14. How many positive integers n satisfy the inequality

$$\left\lceil \frac{n}{101} \right\rceil + 1 > \frac{n}{100}\ ?$$

Recall that $\lceil a \rceil$ is the least integer that is greater than or equal to a.

15. Let $\triangle ABC$ be a triangle with $AB = 7$, $BC = 8$, and $AC = 9$. Point D is on side $\overline{AC}$ such that $\angle CBD$ has measure 45°. What is the length of $\overline{BD}$? Express your answer in the form $m\sqrt{x} - n\sqrt{y}$, where m and n are positive integers and x and y are square-free positive integers.

16. If $-3 \le x < \frac{3}{2}$ and $x \ne 1$, define $C(x) = \frac{x^3}{1-x}$. The real root of the cubic $2x^3 + 3x - 7$ is of the form $pC^{-1}(q)$, where p and q are rational numbers. What is the ordered pair (p, q)? Express your answer using fractions in simplest form.

17. Let f be the function defined by $f(x) = -2\sin(\pi x)$. How many values of x such that $-2 \le x \le 2$ satisfy the equation $f(f(f(x))) = f(x)$?

18. Ranu starts with one standard die on a table. At each step, she rolls all the dice on the table: if all of them show a 6 on top, then she places one more die on the table; otherwise, she does nothing more on this step. After 2013 such steps, let D be the number of dice on the table. What is the expected value (average value) of 6^D?

19. If n is a positive integer, let $\phi(n)$ be the number of positive integers less than or equal to n that are relatively prime to n. Compute the value of the infinite sum

$$\sum_{n=1}^{\infty} \frac{\phi(n)2^n}{9^n - 2^n}.$$

Express your answer as a fraction in simplest form.

20. Let a_0, a_1, a_2, ... be an infinite sequence of real numbers such that $a_0 = \frac{4}{5}$ and

$$a_n = 2a_{n-1}^2 - 1$$

for every positive integer n. Let c be the smallest number such that for every positive integer n, the product of the first n terms satisfies the inequality

$$a_0 a_1 \ldots a_{n-1} \le \frac{c}{2^n}.$$

What is the value of $100c$, rounded to the nearest integer?