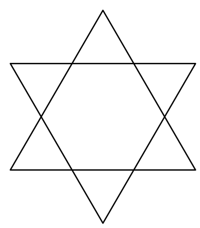
The Advantage Testing Foundation

Math Prize for Girls at MIT

2013 Solutions

Problem 1 The figure below shows two equilateral triangles each with area 1.



The intersection of the two triangles is a regular hexagon. What is the area of the union of the two triangles? Express your answer as a fraction in simplest form.
Answer: $\dfrac{4}{3}$

Solution: The exterior angles of the regular hexagon are each $60°$. So the 6 little triangles sticking out are equilateral triangles. The side length of these little equilateral triangles is $\frac{1}{3}$ that of the 2 big equilateral triangles. Hence the area of each little equilateral triangle is $\frac{1}{9}$.

The union of the 2 big triangles is the disjoint union of 1 big triangle and 3 little triangles. So the total area is

$$1 + 3 \cdot \frac{1}{9} = 1 + \frac{1}{3} = \boxed{\frac{4}{3}}.$$

Problem 2 When the binomial coefficient $\binom{125}{64}$ is written out in base 10, how many zeros are at the rightmost end?
Answer: 0

Solution: The number of zeros at the end of $\binom{125}{64}$ is the number of factors of 10 in $\binom{125}{64}$. In turn, that number is the smaller of the number of factors of 2 in $\binom{125}{64}$ and the number of factors of 5 in $\binom{125}{64}$.

The number of factors of 2 in $\binom{125}{64}$ can be found by looking at the base-2 (binary) representation of 64 and $125 - 64 = 61$. The binary representation of 64 is $(1000000)_2$. The binary representation of 61 is $(0111101)_2$. By Kummer's theorem, the number of factors of 2 in $\binom{125}{64}$ is the number of carries when $(1000000)_2$ and $(0111101)_2$ are added. But $(1000000)_2$ and $(0111101)_2$ have no 1's in common, so there are no carries in their binary addition. Hence the number of factors of 2 in $\binom{125}{64}$ is 0.

Thus the number of factors of 10 in $\binom{125}{64}$ is also $\boxed{0}$.

Problem 3 Let S_1, S_2, ..., S_{125} be 125 sets of 5 numbers each, comprising 625 distinct numbers. Let m_i be the median of S_i. Let M be the median of m_1, m_2, ..., m_{125}. What is the greatest possible number of the 625 numbers that are less than M?
Answer: 436

Solution: Without loss of generality, assume that $m_1 > m_2 > \cdots > m_{125}$. So $M = m_{63}$. Let the set S_i be $\{k_i, \ell_i, m_i, n_i, o_i\}$, where $k_i > \ell_i > m_i > n_i > o_i$. The numbers k_i, ℓ_i, and m_i for $i \le 63$ are each greater than or equal to M. All of the other numbers could be less than M. So the least possible number of the 625 numbers that are greater than or equal to M is $3 \cdot 63 = 189$. Hence the greatest possible number of the 625 numbers that are less than M is $625 - 189$, which is $\boxed{436}$.

Problem 4 The MathMatters competition consists of 10 players P_1, P_2, ..., P_{10} competing in a ladder-style tournament. Player P_{10} plays a game with P_9: the loser is ranked 10th, while the winner plays P_8. The loser of that game is ranked 9th, while the winner plays P_7. They keep repeating this process until someone plays P_1: the loser of that final game is ranked 2nd, while the winner is ranked 1st. How many different rankings of the players are possible?
Answer: 512

Solution: The MathMatters tournament has 9 games. Each game has 2 possible winners. So the tournament has 2^9 possible sequences of winners. Each sequence of winners leads to a unique ranking. So the number of possible rankings is 2^9, which is $\boxed{512}$.

Problem 5 Say that a 4-digit positive integer is *mixed* if it has 4 distinct digits, its leftmost digit is neither the biggest nor the smallest of the 4 digits, and its rightmost digit is not the smallest of the 4 digits. For example, 2013 is mixed. How many 4-digit positive integers are mixed?
Answer: 1680

Solution: The number of sets of 4 digits is $\binom{10}{4}$, which is

$$\frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = 210.$$

Let's fix one choice for the set of 4 digits.

The leftmost digit is neither the biggest nor the smallest, so there are 2 choices for it. (We don't have to worry about starting with a leading zero, because the leftmost digit is not the smallest.) Given that choice, the rightmost digit is neither the smallest nor the leftmost, so there are 2 choices for it. Given those choices, the second digit from the left is neither the leftmost nor the rightmost, so there are 2 choices for it. Finally, given those choices, the third digit from the left has only 1 choice, the remaining digit.

Hence, given a set of 4 digits, there are $2 \cdot 2 \cdot 2 \cdot 1 = 8$ ways to order the digits to form a mixed integer.

So the number of mixed integers is $\binom{10}{4} \cdot 8$, which is $210 \cdot 8$, or $\boxed{1680}$.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 6 Three distinct real numbers form (in some order) a 3-term arithmetic sequence, and also form (in possibly a different order) a 3-term geometric sequence. Compute the greatest possible value of the common ratio of this geometric sequence. Express your answer as a fraction in simplest form.
Answer: $\dfrac{-1}{2}$ or $-\dfrac{1}{2}$

Solution: The geometric sequence has the form a, ar, ar^2, where a and r are real numbers. The three terms are distinct, so $a \neq 0$ and $r \neq 1$.

Because the three numbers form an arithmetic sequence (in some order), one of them is the average of the other two. There are three cases, depending on which number is the average.

Case 1: a is the average of ar and ar^2. In equation form, $2a = ar + ar^2$. So $r^2 + r = 2$. Hence r is -2 or 1. But r can't be 1, so r is -2.

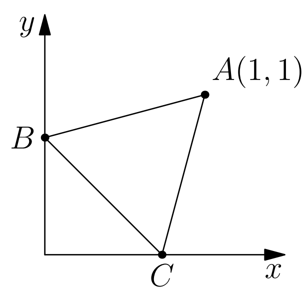
Case 2: ar is the average of a and ar^2. In equation form, $2ar = a + ar^2$. So $r^2 + 1 = 2r$. Hence r is 1. But r can't be 1, so Case 2 is impossible.

Case 3: ar^2 is the average of a and ar. In equation form, $2ar^2 = a + ar$. So $2r^2 = 1 + r$. Hence r is $-\frac{1}{2}$ or 1. But r can't be 1, so r is $-\frac{1}{2}$.

Looking at all three cases, we conclude that r is -2 or $-\frac{1}{2}$. We can achieve $r = -\frac{1}{2}$, since 1, $-\frac{1}{2}$, $\frac{1}{4}$ is a geometric sequence with common ratio $-\frac{1}{2}$ and $-\frac{1}{2}$, $\frac{1}{4}$, 1 is an arithmetic sequence. So the greatest possible value of r is $\boxed{-\dfrac{1}{2}}$.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 7 In the figure below, $\triangle ABC$ is an equilateral triangle.



Point A has coordinates $(1, 1)$, point B is on the positive y-axis, and point C is on the positive x-axis. What is the area of $\triangle ABC$? Express your answer in the form $a\sqrt{n} - b$, where a and b are positive integers and n is a square-free positive integer.

Answer: $2\sqrt{3} - 3$

Solution: We will use complex numbers. For example, the point $A(1, 1)$ in complex form is $1 + i$.

The vector $\overrightarrow{AC}$ in complex form is

$$\overrightarrow{AC} = C - A = C - (1 + i) = C - 1 - i.$$

The vector $\overrightarrow{AB}$ is a 60° clockwise rotation of $\overrightarrow{AC}$. So we have

$$\overrightarrow{AB} = \operatorname{cis}(-60^\circ)\overrightarrow{AC} = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)(C - 1 - i).$$

The complex multiplication on the right expands as follows:

$$\overrightarrow{AB} = \frac{C - 1 - \sqrt{3}}{2} - \frac{(C - 1)\sqrt{3} + 1}{2}i.$$

The vector $\overrightarrow{AB}$ has real part -1. So we get the equation

$$\frac{C - 1 - \sqrt{3}}{2} = -1.$$

Solving for C gives $C = \sqrt{3} - 1$.

We can calculate the squared length of $\overrightarrow{AC}$ as follows:

$$|\overrightarrow{AC}|^2 = |C - 1 - i|^2 = |\sqrt{3} - 2 - i|^2 = (\sqrt{3} - 2)^2 + 1 = 8 - 4\sqrt{3}\,.$$

So the squared side length of equilateral triangle ABC is $8 - 4\sqrt{3}$. Hence the area of $\triangle ABC$ is

$$\frac{(8 - 4\sqrt{3})\sqrt{3}}{4} = (2 - \sqrt{3})\sqrt{3} = \boxed{2\sqrt{3} - 3}\,.$$

Problem 8 Let R be the set of points (x, y) such that x and y are positive, $x + y$ is at most 2013, and

$$\lceil x \rceil \lfloor y \rfloor = \lfloor x \rfloor \lceil y \rceil.$$

Compute the area of set R. Express your answer as a fraction in simplest form. Recall that $\lfloor a \rfloor$ is the greatest integer that is less than or equal to a, and $\lceil a \rceil$ is the least integer that is greater than or equal to a.

Answer: $\dfrac{2013}{2}$

Solution: We may ignore the points when x or y is an integer, since the set of such points has area 0. Because x and y are non-integers, $\lceil x \rceil = \lfloor x \rfloor + 1$ and $\lceil y \rceil = \lfloor y \rfloor + 1$. So the given equation $\lceil x \rceil \lfloor y \rfloor = \lfloor x \rfloor \lceil y \rceil$ becomes

$$(\lfloor x \rfloor + 1) \lfloor y \rfloor = \lfloor x \rfloor (\lfloor y \rfloor + 1)\,.$$

The displayed equation simplifies to $\lfloor y \rfloor = \lfloor x \rfloor$.

Let n be the floor of x (and y). Because $x + y \le 2013$, the value of n is either $0, 1, \ldots, 1005$, or 1006. If $n \le 1005$, then (x, y) is any point in the unit square $n < x < n+1$ and $n < y < n+1$. If $n = 1006$, then (x, y) is any point in the right isosceles triangle given by $1006 < x < 1007$, $1006 < y < 1007$, and $x + y \le 2013$.

The cases $n = 0, 1, \ldots, 1005$ give 1006 unit squares, for an area of 1006. The case $n = 1006$ gives a triangle of area $\frac{1}{2}$. So the total area is $1006\frac{1}{2}$, which is $\boxed{\dfrac{2013}{2}}$.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 9 Let A and B be distinct positive integers such that each has the same number of positive divisors that 2013 has. Compute the least possible value of $|A - B|$.
Answer: 1

Solution: The prime factorization of 2013 is $3 \cdot 11 \cdot 61$. So 2013 has 8 positive divisors.

If A and B are distinct integers, then $|A - B| \geq 1$. Can we find two consecutive integers with exactly 8 positive divisors each?

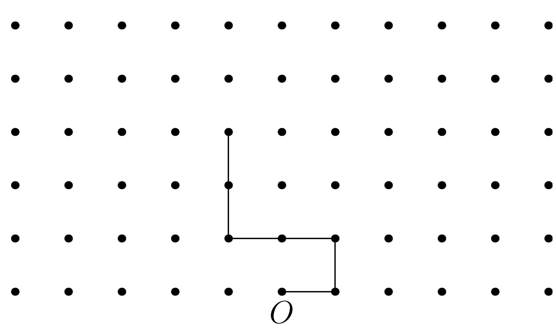
A positive integer with exactly 8 positive divisors has the form p^7 or p^3q or pqr, where p, q, and r are distinct primes. In particular, if q is a prime bigger than 2, then every number of the form $8q$ has exactly 8 positive divisors. Numbers of that form include

$$24, 40, 56, 88, 104, 136.$$

Let's factorize the numbers that are 1 away from these numbers. We see that $105 = 3 \cdot 5 \cdot 7$. So both 104 and 105 have exactly 8 positive divisors. Hence, we can set $A = 104$ and $B = 105$ to get $|A - B| = \boxed{1}$.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 10 The following figure shows a *walk* of length 6:



This walk has three interesting properties:

1. It starts at the origin, labelled O.

2. Each step is 1 unit north, east, or west. There are no south steps.

3. The walk never comes back to a point it has been to.

Let's call a walk with these three properties a *northern walk*. There are 3 northern walks of length 1 and 7 northern walks of length 2. How many northern walks of length 6 are there?
Answer: 239 or 236

Solution: If k is a nonnegative integer, let $N(k)$ be the number of northern walks of length k that end in a *north* step. (For convenience, say that the empty walk of length 0 ends in a north step.) Let $E(k)$ be the number of northern walks of length k that end in an *east* step.

By symmetry, $E(k)$ is also the number of northern walks of length k that end in a *west* step.

For initial conditions, we have $N(0) = 1$ and $E(0) = 0$.

We claim that $E(k + 1) = N(k) + E(k)$. That's because every northern walk of length $k + 1$ that ends in an east step consists of a northern walk of length k that ends in a north or west step followed by a final east step.

Similarly, we claim that $N(k+1) = N(k)+2E(k)$. That's because every northern walk of length $k+1$ that ends in a north step consists of a northern walk of length k (which ends with a north, east, or west step) followed by a final north step.

Using these recurrence relations, we can compute the first few values of N and E.

k	0	1	2	3	4	5	6
$N(k)$	1	1	3	7	17	41	99
$E(k)$	0	1	2	5	12	29	70

From the table above, we see that $N(6) = 99$ and $E(6) = 70$. The number of northern walks of length 6 is $N(6) + 2E(6)$, because every northern walk of length 6 ends in a north, east, or west step. Hence the number of northern walks of length 6 is

$$N(6) + 2E(6) = 99 + 2 \cdot 70 = 99 + 140 = 239.$$

Some students interpreted the problem as excluding walks that went beyond the displayed grid. Under that interpretation, three walks counted above should be excluded: the straight walks of length 6 going in the purely north, east, or west direction. So, under that interpretation, the answer is $239 - 3 = 236$. Our intended answer was 239, but we accepted either $\boxed{\text{239 or 236}}$.

Problem 11 Alice throws two standard dice, with A being the number on her first die and B being the number on her second die. She then draws the line $Ax + By = 2013$. Boris also throws two standard dice, with C being the number on his first die and D being the number on his second die. He then draws the line $Cx + Dy = 2014$. Compute the probability that these two lines are parallel. Express your answer as a fraction in simplest form.

Answer: $\dfrac{43}{648}$

Solution: The slope of the line $Ax + By = 2013$ is $-\frac{A}{B}$. The slope of the line $Cx + Dy = 2014$ is $-\frac{C}{D}$. The lines are parallel if and only if their slopes $-\frac{A}{B}$ and $-\frac{C}{D}$ are the same, which is equivalent to $AD = BC$.

Because A and D are from the set $\{1, 2, 3, 4, 5, 6\}$, the possible values of AD are given by the multiplication table below.

×	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Looking at the table above, we see 5 numbers (1, 9, 16, 25, and 36) that each appear 1 time, 10 numbers (2, 3, 5, 8, 10, 15, 18, 20, 24, and 30) that each appear 2 times, 1 number (4) that appears 3 times, and 2 numbers (6 and 12) that each appear 4 times.

By symmetry, BC has the same distribution.

So the probability that $AD = BC$ is

$$5 \cdot \left(\frac{1}{36}\right)^2 + 10 \cdot \left(\frac{2}{36}\right)^2 + 1 \cdot \left(\frac{3}{36}\right)^2 + 2 \cdot \left(\frac{4}{36}\right)^2,$$

which simplifies to

$$\frac{86}{36^2} = \frac{86}{1296} = \boxed{\frac{43}{648}}.$$

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 12 The rectangular parallelepiped (box) P has some special properties. If one dimension of P were doubled and another dimension were halved, then the surface area of P would stay the same. If instead one dimension of P were tripled and another dimension were divided by 3, then the surface area of P would still stay the same. If the middle (by length) dimension of P is 1, compute the least possible volume of P. Express your answer as a fraction in simplest form.

Answer: $\dfrac{2}{3}$

Solution: Let the dimensions of the parallelepiped P be a, b, and c. The surface area of P is $2(ab + ac + bc)$. If a were doubled and b were halved, then the surface area would be $2(ab+2ac+bc/2)$. Equating the old and new surface area, we get

$$2(ab + ac + bc) = 2(ab + 2ac + bc/2).$$

Simplifying gives $b = 2a$. So the preservation of surface area under a doubling/halving means that one dimension is double another.

If a were tripled and b were divided by 3, then the surface area of P would be $2(ab + 3ac + bc/3)$. Equating the old and new surface area, we get

$$2(ab + bc + ac) = 2(ab + 3ac + bc/3).$$

Simplifying gives $b = 3a$. So the preservation of surface area under a tripling/dividing by 3 means that one dimension is triple another.

We may assume that $a \le b \le c$. Because the middle dimension is 1, we have $b = 1$. Because one dimension is triple another, we have $c \ge 3a$. So the doubling involves b. Hence either $a = \frac{1}{2}$ or $c = 2$.

Suppose that $a = \frac{1}{2}$. Because one dimension is triple another, c is either $\frac{3}{2}$ or 3. Either way, the volume of P is at least $\frac{3}{4}$.

Suppose that $c = 2$. Because one dimension is triple another, a is either $\frac{1}{3}$ or $\frac{2}{3}$. Either way, the volume of P is at least $\frac{2}{3}$.

Combining the two cases, we see that the volume of P is at least $\frac{2}{3}$. We can achieve that bound by setting $a = \frac{1}{3}$, $b = 1$, and $c = 2$. So the least possible volume of P is $\boxed{\dfrac{2}{3}}$.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 13 Each of n boys and n girls chooses a random number from the set $\{1, 2, 3, 4, 5\}$, uniformly and independently. Let p_n be the probability that every boy chooses a different number than every girl. As n approaches infinity, what value does $\sqrt[n]{p_n}$ approach? Express your answer as a fraction in simplest form.

Answer: $\dfrac{6}{25}$

Solution: The probability that each of the n boys chooses a number from the set $\{1, 2, 3\}$ and each of the n girls chooses a number from $\{4, 5\}$ is

$$\left(\frac{3}{5}\right)^n \cdot \left(\frac{2}{5}\right)^n = \left(\frac{6}{25}\right)^n .$$

So $p_n \geq (6/25)^n$.

More generally, let S be a subset of $\{1, 2, 3, 4, 5\}$. Let A_S be the event that each of the n boys chooses a number from S and each of the n girls chooses a number not from S. The probability of A_S is

$$\left(\frac{|S|}{5}\right)^n \cdot \left(\frac{5-|S|}{5}\right)^n = \left(\frac{|S|(5-|S|)}{25}\right)^n .$$

The numerator $|S|(5 - |S|)$ is at most 6, so the probability of A_S is at most $(6/25)^n$.

The event "every boy chooses a different number than every girl" is the union of the events A_S over all subsets S of $\{1, 2, 3, 4, 5\}$. In other words,

$$p_n = \Pr[\bigcup_S A_S].$$

Because the probability of a union is at most the sum of the individual probabilities, we have

$$p_n \leq \sum_S \Pr[A_S].$$

In the previous paragraph, we showed that $\Pr[A_S]$ is at most $(6/25)^n$, so

$$p_n \leq \sum_S \left(\frac{6}{25}\right)^n .$$

Because S ranges over the 32 subsets of $\{1, 2, 3, 4, 5\}$, we have

$$p_n \leq 32 \left(\frac{6}{25}\right)^n .$$

Our previous work has shown that $(6/25)^n \le p_n \le 32(6/25)^n$. Taking the nth root shows that $\frac{6}{25} \le \sqrt[n]{p_n} \le \frac{6}{25}\sqrt[n]{32}$. As n approaches infinity, $\sqrt[n]{32}$ approaches 1. So $\sqrt[n]{p_n}$ approaches $\boxed{\dfrac{6}{25}}$.

Problem 14 How many positive integers n satisfy the inequality

$$\left\lceil \frac{n}{101} \right\rceil + 1 > \frac{n}{100}\,?$$

Recall that $\lceil a \rceil$ is the least integer that is greater than or equal to a.
Answer: 15,049

Solution: Every positive integer n can be written uniquely in the form $101q - r$, where q is a positive integer and r is a nonnegative integer such that $r \le 100$. Here $q = \lceil n/101 \rceil$. The given inequality becomes

$$q + 1 > \frac{101q - r}{100}\,.$$

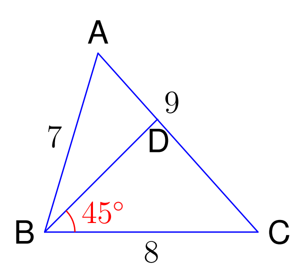
Solving for q gives $q < 100 + r$. Because q is an integer, that inequality is equivalent to $q \le 99 + r$. So for any given r, there are $99 + r$ choices for q. Hence the total number of choices is

$$\sum_{r=0}^{100}(99 + r) = 101(99) + \sum_{r=0}^{100} r = 9999 + 5050 = \boxed{15049}.$$

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 15 Let $\triangle ABC$ be a triangle with $AB = 7$, $BC = 8$, and $AC = 9$. Point D is on side $\overline{AC}$ such that $\angle CBD$ has measure 45°. What is the length of $\overline{BD}$? Express your answer in the form $m\sqrt{x} - n\sqrt{y}$, where m and n are positive integers and x and y are square-free positive integers.
Answer: $40\sqrt{2} - 16\sqrt{10}$

Solution: The diagram below shows the triangle ABC and the given information.



By the Law of Cosines, we have

$$7^2 = 8^2 + 9^2 - 2 \cdot 8 \cdot 9 \cos C.$$

Solving for cosine gives

$$\cos C = \frac{8^2 + 9^2 - 7^2}{2 \cdot 8 \cdot 9} = \frac{96}{2 \cdot 8 \cdot 9} = \frac{6}{9} = \frac{2}{3}\,.$$

Hence the sine is

$$\sin C = \sqrt{1 - \cos^2 C} = \sqrt{1 - \left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}\,.$$

Because the angles of triangle BCD add to 180°, we have

$$\angle BDC = 180^\circ - 45^\circ - C = 135^\circ - C.$$

By the sine subtraction law, we have

$$\sin \angle BDC = \sin(135^\circ - C) = \frac{\sqrt{2}}{2} \cdot \frac{2}{3} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{5}}{3} = \frac{\sqrt{10} + 2\sqrt{2}}{6}\,.$$

By the Law of Sines, we have

$$\frac{BD}{\sin C} = \frac{8}{\sin \angle BDC} = \frac{8 \cdot 6}{\sqrt{10} + 2\sqrt{2}} = 24\sqrt{10} - 48\sqrt{2}.$$

Solving for BD gives

$$BD = (\sin C)(24\sqrt{10} - 48\sqrt{2}\,) = \frac{\sqrt{5}}{3}(24\sqrt{10} - 48\sqrt{2}\,) = \boxed{40\sqrt{2} - 16\sqrt{10}}.$$

Problem 16 If $-3 \le x < \frac{3}{2}$ and $x \ne 1$, define $C(x) = \frac{x^3}{1-x}$. The real root of the cubic $2x^3 + 3x - 7$ is of the form $pC^{-1}(q)$, where p and q are rational numbers. What is the ordered pair (p, q)? Express your answer using fractions in simplest form.

Answer: $\left(\dfrac{7}{3}, \dfrac{27}{98}\right)$

Solution: Let x be the real root of $2x^2 + 3x - 7$. By the Rational Root theorem, we see that x is irrational.

According to the statement of the problem, $x = pC^{-1}(q)$, where p and q are rational. Solving for q, we find that

$$q = C\left(\frac{x}{p}\right) = \frac{(x/p)^3}{1 - x/p} = \frac{x^3}{p^3 - p^2 x} = \frac{7 - 3x}{2p^3 - 2p^2 x}\,.$$

Clearing fractions gives

$$2p^3 q - 2p^2 q x = 7 - 3x.$$

Grouping the x terms together, we get

$$(2p^2 q - 3)x = 2p^3 q - 7.$$

Because x is irrational, it follows that $2p^2 q = 3$ and $2p^3 q = 7$. Dividing one equation by the other shows that

$$p = \frac{2p^2 q}{2p^3 q} = \frac{7}{3}\,.$$

Hence

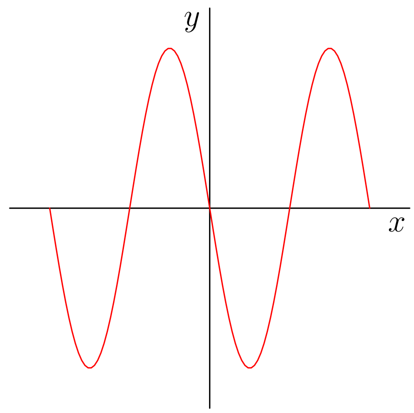
$$q = \frac{3}{2p^2} = \frac{3}{2(7/3)^2} = \frac{3^3}{2(7)^2} = \frac{27}{2 \cdot 49} = \frac{27}{98}\,.$$

Therefore, the pair (p, q) is $\boxed{\left(\frac{7}{3}, \frac{27}{98}\right)}$.

Note: The function C^{-1} is called the "curly root" function by Dan Kalman in his book *Uncommon Mathematical Excursions: Polynomia and Related Realms*. He uses curly roots as an alternative method for solving cubic equations.

Problem 17 Let f be the function defined by $f(x) = -2\sin(\pi x)$. How many values of x such that $-2 \le x \le 2$ satisfy the equation $f(f(f(x))) = f(x)$?
Answer: 61

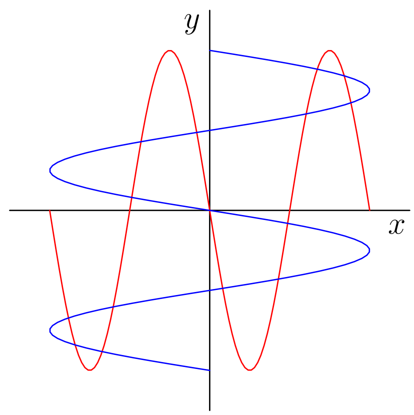
Solution: The graph of f on the interval $[-2, 2]$ is shown below.



As we can see, the equation $f(x) = 0$ has 5 solutions in $[-2, 2]$. If y is a fixed nonzero number such that $-2 < y < 2$, then the equation $f(x) = y$ has 4 solutions in $[-2, 2]$.

Let's enumerate the solutions of $f(f(f(x))) = f(x)$ in $[-2, 2]$. Let $y = f(x)$. Then the equation becomes $f(f(y)) = y$.

So we need to enumerate the solutions of $f(f(y)) = y$ in $[-2, 2]$. Let $x = f(y)$. Then $y = f(x)$ and $x = f(y)$. The graph of $y = f(x)$ is the graph of f, as shown above. The graph of $x = f(y)$ is the inverse graph, the reflection of $y = f(x)$ around the line $y = x$. The diagram below shows the two graphs, restricted to x and y in $[-2, 2]$.



As we can see, ignoring the origin, the two graphs intersect at 4 points in the first quadrant, 3 points in the second quadrant, 4 points in the third quadrant, and 3 points in the fourth quadrant. Altogether, there are 14 intersection points, not counting the origin. So there are 14 nonzero solutions to $f(f(y)) = y$ in $[-2, 2]$. Neither 2 nor -2 is a solution.

As we just discovered, there are 14 nonzero solutions to $f(f(y)) = y$ in the open interval $(-2, 2)$. As we mentioned earlier, for every nonzero y in $(-2, 2)$, there are 4 solutions to $f(x) = y$ in $[-2, 2]$. So there are $14 \cdot 4 = 56$ solutions to $f(f(f(x))) = f(x)$ and $f(x) \neq 0$ in $[-2, 2]$. Plus the 5 solutions to $f(x) = 0$ in $[-2, 2]$ give 5 solutions to $f(f(f(x))) = f(x)$ and $f(x) = 0$ in $[-2, 2]$. Altogether, the number of solutions to $f(f(f(x))) = f(x)$ in $[-2, 2]$ is $56 + 5$, which is $\boxed{61}$.

Problem 18 Ranu starts with one standard die on a table. At each step, she rolls all the dice on the table: if all of them show a 6 on top, then she places one more die on the table; otherwise, she does nothing more on this step. After 2013 such steps, let D be the number of dice on the table. What is the expected value (average value) of 6^D?
Answer: 10,071

Solution: For every nonnegative integer n, let D_n be the number of dice on the table after n steps. At the start, $D_0 = 1$. Given D_n, we are told that D_{n+1} is $D_n + 1$ with probability 6^{-D_n} and is D_n otherwise. Hence the expected value of $6^{D_{n+1}}$, given D_n, is

$$E(6^{D_{n+1}} \mid D_n) = 6^{-D_n} \cdot 6^{D_n+1} + \left(1 - 6^{-D_n}\right) \cdot 6^{D_n} = 6 + 6^{D_n} - 1 = 6^{D_n} + 5.$$

Averaging over D_n shows that

$$E(6^{D_{n+1}}) = E(E(6^{D_{n+1}} \mid D_n)) = E(6^{D_n} + 5) = E(6^{D_n}) + 5.$$

Each expected value is 5 more than the previous one So we have

$$E(6^{D_{2013}}) = E(6^{D_0}) + 2013 \cdot 5 = E(6^1) + 10065 = 6 + 10065 = \boxed{10071}.$$

Note: This problem is related to the "approximate counting" randomized algorithm in computer science that is used to count a large number of events with a small amount of memory.

Problem 19 If n is a positive integer, let $\phi(n)$ be the number of positive integers less than or equal to n that are relatively prime to n. Compute the value of the infinite sum

$$\sum_{n=1}^{\infty} \frac{\phi(n) 2^n}{9^n - 2^n}.$$

Express your answer as a fraction in simplest form.
Answer: $\frac{18}{49}$

Solution: Let $x = \frac{2}{9}$. Then the given sum can be rewritten as

$$\sum_{n=1}^{\infty} \phi(n) \frac{x^n}{1 - x^n} .$$

By geometric series, we have

$$\frac{x^n}{1 - x^n} = \sum_{k=1}^{\infty} x^{kn}.$$

By setting $m = kn$, we can rewrite the last sum as

$$\frac{x^n}{1 - x^n} = \sum_{\substack{m=1 \\ n|m}}^{\infty} x^m,$$

where m ranges over all positive multiples of n. So our original sum becomes the double sum

$$\sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ n|m}}^{\infty} \phi(n) x^m.$$

By switching the order of the sums, we get

$$\sum_{m=1}^{\infty} \sum_{\substack{n=1 \\ n|m}}^{\infty} \phi(n) x^m.$$

For every fixed positive integer m, we claim that

$$\sum_{\substack{n=1 \\ n|m}}^{\infty} \phi(n) = m.$$

If d is a positive divisor of m, then the number of positive integers x less than or equal to m such that $\gcd(x, m) = d$ is $\phi(m/d)$. Summing over all d shows that

$$\sum_{\substack{d \\ d|m}} \phi(m/d) = m.$$

Setting $n = m/d$ proves the claim.

We can use the claim of the last paragraph to simplify the double sum before that:

$$\sum_{m=1}^{\infty} \sum_{\substack{n=1 \\ n|m}}^{\infty} \phi(n) x^m = \sum_{m=1}^{\infty} m x^m.$$

The sum on the right can be evaluated in standard ways:

$$\sum_{m=1}^{\infty} m x^m = \sum_{m=1}^{\infty} \sum_{k=1}^{m} x^m = \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} x^m = \sum_{k=1}^{\infty} \frac{x^k}{1-x} = \frac{x}{(1-x)^2}\,.$$

Plugging in $x = \frac{2}{9}$, we get

$$\frac{2/9}{(1-2/9)^2} = \frac{2 \cdot 9}{7^2} = \boxed{\frac{18}{49}}.$$

Note: The infinite sum in the statement of the problem is an example of a Lambert series.

Problem 20 Let a_0, a_1, a_2, ... be an infinite sequence of real numbers such that $a_0 = \frac{4}{5}$ and

$$a_n = 2a_{n-1}^2 - 1$$

for every positive integer n. Let c be the smallest number such that for every positive integer n, the product of the first n terms satisfies the inequality

$$a_0 a_1 \dots a_{n-1} \le \frac{c}{2^n}.$$

What is the value of $100c$, rounded to the nearest integer?
Answer: 167

Solution: The recurrence $a_n = 2a_{n-1}^2 - 1$ resembles the double-angle identity $\cos 2\theta = 2\cos^2\theta - 1$. With that in mind, we shall set

$$\theta = \cos^{-1} a_0 = \cos^{-1} \frac{4}{5}\,.$$

Then we have

$$a_1 = 2a_0^2 - 1 = 2\cos^2\theta - 1 = \cos 2\theta.$$

In the same way, we find that

$$a_k = \cos 2^k \theta.$$

We're interested in the product

$$\cos\theta \cos 2\theta \cos 4\theta \ldots \cos 2^{n-1}\theta.$$

Using the double-angle identity $\sin 2\theta = 2\sin\theta\cos\theta$, we get the extended identity

$$\sin\theta\cos\theta\cos 2\theta\cos 4\theta \ldots \cos 2^{n-1}\theta = \sin 2^n\theta.$$

Because $a_k = \cos 2^k\theta$, we have

$$2^n(\sin\theta)a_0a_1a_2\ldots a_{n-1} = \sin 2^n\theta.$$

Because $\cos\theta = \frac{4}{5}$, we have $\sin\theta = \frac{3}{5}$. Therefore

$$a_0a_1a_2\ldots a_{n-1} = \frac{5\sin 2^n\theta}{3\cdot 2^n}.$$

Because $\sin 2^n\theta \le 1$, we have

$$a_0a_1a_2\ldots a_{n-1} \le \frac{5}{3\cdot 2^n}.$$

Hence $c \le \frac{5}{3}$.

If $\sin 2^n\theta$ gets arbitrarily close to 1, then the analysis above would show that $c = \frac{5}{3}$. Does $\sin 2^n\theta$ get arbitrarily close to 1? Probably, but I don't know how to prove it. (It's related to the binary representation of θ.) Nevertheless, we can check by computer that

$$\sin 2^{36}\theta > 0.999.$$

Then the analysis above shows that

$$c \ge \frac{5}{3}\sin 2^{36}\theta > \frac{5}{3}\cdot 0.999 = 1.665.$$

So $100c$ is greater than 166.5 and at most $\frac{500}{3} = 166\frac{2}{3}$. Hence $100c$ rounded to the nearest integer is $\boxed{167}$.