THE ADVANTAGE TESTING FOUNDATION

MATH PRIZE for GIRLS at MIT

2014 OLYMPIAD SOLUTIONS

Problem 1 Say that a convex quadrilateral is *tasty* if its two diagonals divide the quadrilateral into four nonoverlapping similar triangles. Find all tasty convex quadrilaterals. Justify your answer.

Solution: Let $ABCD$ be a tasty quadrilateral. Let P be the point where the diagonals $\overline{AC}$ and $\overline{BD}$ intersect. See the diagram below.
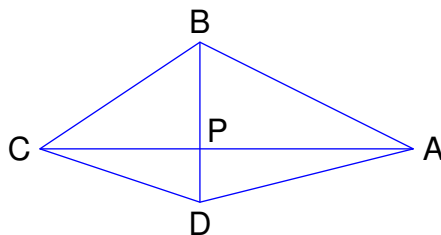
Consider the angle $\angle APB$. It is the sum of the two remote angles $\angle PBC$ and $\angle PCB$. So it is bigger than both remote angles. Because triangles APB and BPC are similar (in some order), it follows that $\angle APB = \angle BPC$. Because $\angle APB$ and $\angle BPC$ form a line, both are right angles. Similarly, $\angle CPD$ and $\angle DPA$ are right angles. So the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular.

Among the lengths AP, BP, CP, and DP, assume without loss of generality that AP is the smallest. By scaling, we may assume that $AP = 1$. Let $r = BP$. Because AP is smallest, $r \geq 1$. Because $\triangle APB$ and $\triangle DPA$ are similar (in some order), DP is either r or $1/r$. Because AP is smallest, $DP = r$.

By SAS congruence, $\triangle APB$ and $\triangle APD$ are congruent, so $AB = AD$. By SAS congruence, $\triangle CPB$ and $\triangle CPD$ are congruent, so $CB = CD$. Hence $ABCD$ is a kite.

Because triangles BPC and APB are similar (in some order), CP is either 1 or r^2.

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Let's first handle the case $CP = 1$. Then the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular bisectors of each other. So $ABCD$ is a rhombus.

Next, let's handle the case $CP = r^2$. Then $\triangle APB$ and $\triangle BPC$ are similar (in that order), so $\angle PAB = \angle PBC$. Because $\angle PBA$ and $\angle PAB$ are complements, so are $\angle PBA$ and $\angle PBC$. Hence $\angle ABC$ is a right angle. Similarly, $\angle ADC$ is a right angle. As we showed above, $ABCD$ is a kite, so $ABCD$ is a right kite, a kite with right angles at two opposite angles. (A right kite is the same as a cyclic kite.)

So every tasty quadrilateral is a rhombus or a right kite. We will now show the converse: every rhombus or right kite is tasty.

Consider a rhombus. Its sides are of equal length. Because it is a parallelogram, its diagonals bisect each other. So, by SSS congruence, its diagonals divide it into four congruent triangles. Hence every rhombus is tasty.

Next consider a right kite. One diagonal divides the kite into two congruent right triangles. The other diagonal divides each of these right triangles into two similar triangles. (In a right triangle, the altitude to the hypotenuse divides the triangle into two similar triangles.) So the two diagonals divide the kite into four similar triangles. Hence every right kite is tasty.

We have showed that a convex quadrilateral is tasty if and only if it is a rhombus or a right kite. In other words, the set of tasty quadrilaterals is the union of the set of rhombuses and the set of right kites.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 2 Let f be the function defined by $f(x) = 4x(1 - x)$. Let n be a positive integer. Prove that there exist distinct real numbers x_1, x_2, ..., x_n such that $x_{i+1} = f(x_i)$ for each integer i with $1 \leq i \leq n - 1$, and such that $x_1 = f(x_n)$.

Solution: Given an angle θ, note that

$$f(\sin^2 \theta) = 4 \sin^2 \theta (1 - \sin^2 \theta) = 4 \sin^2 \theta \cos^2 \theta = (2 \sin \theta \cos \theta)^2 = \sin^2 2\theta.$$

With that identity in mind, let

$$x_i = \sin^2(2^{i-1}\alpha),$$

where α is an angle to be specified later. Note that

$$x_{i+1} = \sin^2(2^i \alpha) = \sin^2(2 \cdot 2^{i-1}\alpha) = f(\sin^2(2^{i-1}\alpha)) = f(x_i),$$

as desired. With foresight, let's choose

$$\alpha = \frac{\pi}{2^n + 1}.$$

We have

$$x_1 = \sin^2 \alpha = \sin^2(\pi - \alpha) = \sin^2(2^n \alpha) = x_{n+1} = f(x_n),$$

as desired.

We need to show that x_1, ..., x_n are distinct real numbers. Because $0 < \alpha < \frac{\pi}{2^n}$, we have

$$0 < \alpha < 2\alpha < 2^2\alpha < \cdots < 2^{n-1}\alpha < \frac{\pi}{2}.$$

Because $\sin$ (and hence $\sin^2$) is strictly increasing on the interval $[0, \frac{\pi}{2}]$, we have

$$x_1 < x_2 < \cdots < x_n.$$

Hence x_1, ..., x_n are distinct.

Alternative Solution (sketch): The case $n = 1$ is trivial, so assume that $n \geq 2$. Let f^i be the ith iterate of f. Let b be the smallest positive number such that $f^n(b) = 0$; it's easy to see that $f^{n-1}(b) = 1$ and $f^{n-2}(b) = \frac{1}{2}$. Let a be the smallest positive number such that $f^n(a) = a$; it's not hard to show that $a < b$. Because $f^{n-2}(b) = \frac{1}{2}$, it follows that $f^i(a) \leq \frac{1}{2}$ for all $i \leq n - 2$. Note that $f(x) > x$ for all $0 < x < \frac{3}{4}$. So $a < f(a) < f^2(a) < \cdots < f^{n-1}(a)$. Let $x_i = f^{i-1}(a)$. Then $x_1 < x_2 < \cdots < x_n$. So the x_i are distinct real numbers that satisfy the conditions of the problem.

Problem 3 Say that a positive integer is *sweet* if it uses only the digits 0, 1, 2, 4, and 8. For instance, 2014 is sweet. There are sweet integers whose squares are sweet: some examples (not necessarily the smallest) are 1, 2, 11, 12, 20, 100, 202, and 210. There are sweet integers whose cubes are sweet: some examples (not necessarily the smallest) are 1, 2, 10, 20, 200, 202, 281, and 2424. Prove that there exists a sweet positive integer n whose square and cube are both sweet, such that the sum of all the digits of n is 2014.

Solution: We will define n to be

$$n = 2 \sum_{i=1}^{1007} 100^{a_i},$$

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where a_1, a_2, ..., a_{1007} are distinct nonnegative integers to be chosen later. Because the a_i are distinct, n uses only the digits 0 and 2. So n is sweet. The number of 2's in n is 1007, so the digit sum of n is 2014, as desired.

We now consider n^2. We need to square a sum:

$$(\sum_i x_i)^2 = \sum_{i,j} x_i x_j = \sum_i x_i^2 + 2\sum_{i<j} x_i x_j.$$

Hence n^2 is

$$n^2 = 4\sum_i 100^{2a_i} + 8\sum_{i<j} 100^{a_i+a_j}.$$

We will choose the a_i so that $a_i + a_j$ is different for each pair $i \leq j$. In that case, n^2 uses only the digits 0, 4, and 8. So n^2 is sweet.

We now consider n^3. We need to cube a sum:

$$(\sum_i x_i)^3 = \sum_{i,j,k} x_i x_j x_k = \sum_i x_i^3 + 3\sum_{i\neq j} x_i^2 x_j + 6\sum_{i<j<k} x_i x_j x_k.$$

Hence n^3 is

$$n^3 = 8\sum_i 100^{3a_i} + 24\sum_{i\neq j} 100^{2a_i+a_j} + 48\sum_{i<j<k} 100^{a_i+a_j+a_k}.$$

We will choose the a_i so that $a_i+a_j+a_k$ is different for each $i \leq j \leq k$. In that case, n^3 uses only the digits 0, 2, 4, and 8. (The two-digit coefficients such as 24 and 48 won't overlap with other coefficients because we used base 100.) So n^3 is sweet.

We now choose the a_i to satisfy the distinctness conditions above. We can set $a_i = 4^i$ for all i. This choice works because the base-4 representation of an integer is unique, and a sum such as $a_i + a_j$ or $a_i + a_j + a_k$ has at most 3 occurrences of the same power of 4. Alternatively, we could bound a_i by a polynomial (of degree 5) in i by using a greedy approach: choose a_1, a_2, ..., in that order by setting each a_i to the smallest nonnegative integer consistent with the previous choices.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 4 Let n be a positive integer. A 4-by-n rectangle is divided into $4n$ unit squares in the usual way. Each unit square is colored black or white. Suppose that every white unit square shares an edge with at least one black unit square. Prove that there are at least n black unit squares.

Solution: The 4-by-n rectangle consists of n columns, each with 4 unit squares. If some column is all white, then (to touch its white squares) its neighboring columns must have at least 4 black squares in all. In particular, for each all-white column, either its left neighbor has at least 3 black squares or its right neighbor has at least 2 black squares (or both).

We need to prove that there are at least n black squares. To do so, we will move the black squares around so that there is at least one black square in each column. (Such movements might destroy the property that each white square is next to a black square, but that's okay. All that matters is that the number of black squares stays the same.)

Each all-white column will take a black square from its left neighbor (provided this neighbor started with at least 3 black squares); otherwise it will take a black square from its right neighbor (which started with at least 2 black squares).

The claim is that each column now has at least one black square. If a column started with 0 black squares, then it took a black square from one of its neighbors. If a column started with 1 black square, then it kept its black square. If a column started with 2 black squares, then it may have given 1 black square to its left neighbor, but it will still have at least 1 black square. If a column started with 3 or 4 black squares, then it may have given 1 black square each to its left and right neighbors, but it will still have at least 1 black square. So every column has at least 1 black square after the movements.

Because each column ends with at least 1 black square, the number of black squares at the end is at least n. But the number of black squares never changed. So the original coloring has at least n black squares.