Advantage Testing Foundation

The Sixth Annual Math Prize for Girls

Saturday, September 27, 2014

TEST BOOKLET

Test Version A

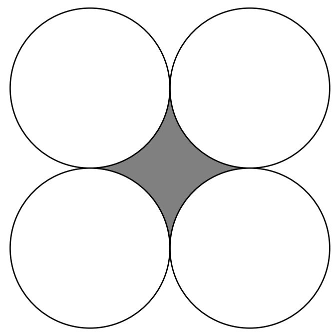
DIRECTIONS

1. Do not open this test until your proctor instructs you to.
2. Fill out the top of your answer sheet. Your student number is on your name tag. Your test version (A, B, C, or D) is above on this page.
3. You may use pencils, pens, and erasers. You may also use the scratch paper that we provide. You may NOT use calculators, rulers, protractors, compasses, graph paper, books, or slide rules.
4. Write your final answers on the answer sheet. When a problem specifies a particular form for the answer, write your answer in that form. Write your answers clearly. We will collect only the answer sheet from you.

NOTES

1. This test contains 20 problems. You will have 150 minutes (2.5 hours) to take the test. Your score will be the number of correct answers.
2. Figures are not necessarily drawn to scale.
3. Good luck!

1. The four congruent circles below touch one another and each has radius 1.



What is the area of the shaded region? Express your answer in terms of π.

2. Let x_1, x_2, $\ldots$, x_{10} be 10 numbers. Suppose that $x_i + 2x_{i+1} = 1$ for each i from 1 through 9. What is the value of $x_1 + 512x_{10}$?

3. Four different positive integers less than 10 are chosen randomly. What is the probability that their sum is odd? Express your answer as a fraction in simplest form.

4. Say that an integer A is *yummy* if there exist several consecutive integers (including A) that add up to 2014. What is the smallest yummy integer?

5. Say that an integer $n \geq 2$ is *delicious* if there exist n positive integers adding up to 2014 that have distinct remainders when divided by n. What is the smallest delicious integer?

6. There are N students in a class. Each possible nonempty group of students selected a positive integer. All of these integers are distinct and add up to 2014. Compute the greatest possible value of N.

7. If x is a real number and k is a nonnegative integer, recall that the binomial coefficient $\binom{x}{k}$ is defined by the formula

$$\binom{x}{k} = \frac{x(x-1)(x-2)\dots(x-k+1)}{k!}\,.$$

Compute the value of

$$\frac{\binom{1/2}{2014} \cdot 4^{2014}}{\binom{4028}{2014}}\,.$$

Express your answer as a fraction in simplest form.

8. A triangle has sides of length $\sqrt{13}$, $\sqrt{17}$, and $2\sqrt{5}$. Compute the area of the triangle.

9. Let abc be a three-digit prime number whose digits satisfy $a < b < c$. The difference between every two of the digits is a prime number too. What is the sum of all the possible values of the three-digit number abc?

10. An ant is on one face of a cube. At every step, the ant walks to one of its four neighboring faces with equal probability. What is the expected (average) number of steps for it to reach the face opposite its starting face?

11. Let R be the set of points (x, y) such that $\lfloor x^2 \rfloor = \lfloor y \rfloor$ and $\lfloor y^2 \rfloor = \lfloor x \rfloor$. Compute the area of region R. Express your answer in the form $a - b\sqrt{c}$, where a and b are positive integers and c is a square-free positive integer. Recall that $\lfloor z \rfloor$ is the greatest integer that is less than or equal to z.

12. Let B be a $1 \times 2 \times 4$ box (rectangular parallelepiped). Let R be the set of points that are within distance 3 of some point in B. (Note that R contains B.) What is the volume of R? Express your answer in terms of π.

13. Deepali has a bag containing 10 red marbles and 10 blue marbles (and nothing else). She removes a random marble from the bag. She keeps doing so until all of the marbles remaining in the bag have the same color. Compute the probability that Deepali ends with exactly 3 marbles remaining in the bag. Express your answer as a fraction in simplest form.

14. A triangle has area 114 and sides of integer length. What is the perimeter of the triangle?

15. There are two math exams called A and B. 2014 students took the A exam and/or the B exam. Each student took one or both exams, so the total number of exam papers was between 2014 and 4028, inclusive. The score for each exam is an integer from 0 through 40. The average score of all the exam papers was 20. The grade for a student is the best score from one or both exams that she took. The average grade of all 2014 students was 14. Let G be the *greatest* possible number of students who took both exams. Let L be the *least* possible number of students who took both exams. Compute $G - L$.

16. If $\sin x + \sin y = \frac{96}{65}$ and $\cos x + \cos y = \frac{72}{65}$, then what is the value of $\tan x + \tan y$? Express your answer as a fraction in simplest form.

17. Let ABC be a triangle. Points D, E, and F are respectively on the sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ of $\triangle ABC$. Suppose that

$$\frac{AE}{AC} = \frac{CD}{CB} = \frac{BF}{BA} = x$$

for some x with $\frac{1}{2} < x < 1$. Segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ cut the triangle into 7 nonoverlapping regions: 4 triangles and 3 quadrilaterals. The total area of the 4 triangles equals the total area of the 3 quadrilaterals. Compute the value of x. Express your answer in the form $\frac{k-\sqrt{m}}{n}$, where k and n are positive integers and m is a square-free positive integer.

18. For how many integers k such that $0 \le k \le 2014$ is it true that the binomial coefficient $\binom{2014}{k}$ is a multiple of 4?

19. Let n be a positive integer. Let (a, b, c) be a random ordered triple of nonnegative integers such that $a + b + c = n$, chosen uniformly at random from among all such triples. Let M_n be the expected value (average value) of the largest of a, b, and c. As n approaches infinity, what value does $\frac{M_n}{n}$ approach? Express your answer as a fraction in simplest form.

20. How many complex numbers z such that $|z| < 30$ satisfy the equation

$$e^z = \frac{z-1}{z+1}\,?$$