

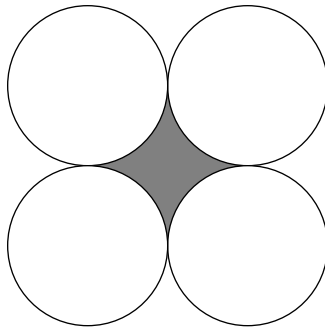
THE ADVANTAGE TESTING FOUNDATION



MATH PRIZE *for* GIRLS *at* MIT

2014 SOLUTIONS

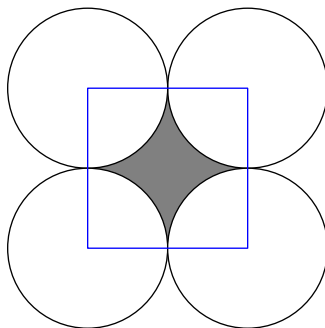
Problem 1 The four congruent circles below touch one another and each has radius 1.



What is the area of the shaded region? Express your answer in terms of π .

Answer: $4 - \pi$ or $-\pi + 4$

Solution: Join the centers of the 4 circles as in the figure below.



The centers of the circles are the vertices of a square. The square has side length $1 + 1 = 2$. So the square has area $2^2 = 4$. The unshaded portion of the square consists of 4 quarter circles, each of radius 1. These 4 quarter circles form a whole circle of radius 1. So the unshaded portion of the square has area π . Hence the shaded region has area $\boxed{4 - \pi}$.

Problem 2 Let x_1, x_2, \dots, x_{10} be 10 numbers. Suppose that $x_i + 2x_{i+1} = 1$ for each i from 1 through 9. What is the value of $x_1 + 512x_{10}$?

Answer: 171

Solution: To simplify the given equation $x_i + 2x_{i+1} = 1$, let $y_i = x_i - \frac{1}{3}$. Then the equation becomes $y_i + 2y_{i+1} = 0$, or $y_i = -2y_{i+1}$. Unwinding, we get

$$y_1 = -2y_2 = (-2)^2y_3 = (-2)^3y_4 = \dots = (-2)^9y_{10} = -512y_{10}.$$

So $y_1 + 512y_{10} = 0$. Going back to x_i , we get

$$x_1 + 512x_{10} = \left(y_1 + \frac{1}{3}\right) + 512\left(y_{10} + \frac{1}{3}\right) = (y_1 + 512y_{10}) + \frac{1}{3} + \frac{512}{3} = \boxed{171}.$$

Problem 3 Four different positive integers less than 10 are chosen randomly. What is the probability that their sum is odd? Express your answer as a fraction in simplest form.

Answer: $\frac{10}{21}$

Solution: We are choosing 4 integers from 1 to 9. The number of ways of doing so is

$$\binom{9}{4} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{24} = 9 \cdot 7 \cdot 2 = 126.$$

To make the sum of the 4 integers odd, we will choose either 3 odd and 1 even (case 1) or 1 odd and 3 even (case 2). The number of ways to satisfy case 1 is

$$\binom{5}{3} \binom{4}{1} = 10 \cdot 4 = 40.$$

The number of ways to satisfy case 2 is

$$\binom{5}{1} \binom{4}{3} = 5 \cdot 4 = 20.$$

So the total number of successful choices is $40 + 20 = 60$. Hence the probability of success is

$$\frac{60}{126} = \frac{30}{63} = \boxed{\frac{10}{21}}.$$

Problem 4 Say that an integer A is *yummy* if there exist several consecutive integers (including A) that add up to 2014. What is the smallest yummy integer?

Answer: -2013

Solution: Here is a sequence of consecutive integers that add up to 2014:

$$-2013, -2012, \dots, -1, 0, 1, \dots, 2012, 2013, 2014.$$

So -2013 is yummy.

Assume there is a yummy integer less than -2013 . Then there is a sequence of consecutive integers (including at least one less than -2013) that add up to 2014. Let A be the least integer in the sequence, so $A < -2013$. Because the sum of the sequence is nonnegative, it includes the numbers

$$A, \dots, -1, 0, 1, \dots, -A.$$

Because the sum of the sequence is positive, besides the numbers above, it includes $-A + 1$. But

$$-A + 1 > 2013 + 1 = 2014.$$

So the sum of the sequence exceeds 2014, which is a contradiction. Hence there is no yummy integer less than -2013 .

Therefore the least yummy integer is $\boxed{-2013}$.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 5 Say that an integer $n \geq 2$ is *delicious* if there exist n positive integers adding up to 2014 that have distinct remainders when divided by n . What is the smallest delicious integer?

Answer: 4

Solution: Is 2 delicious? The two remainders mod 2 are 0 and 1. Their sum is 1 mod 2. But 2014 isn't 1 mod 2. So 2 isn't delicious.

Is 3 delicious? The three remainders mod 3 are 0, 1, and 2. Their sum is 0 mod 3. But 2014 isn't 0 mod 3. So 3 isn't delicious.

Is 4 delicious? The four remainders mod 4 are 0, 1, 2, and 3. Their sum is 2 mod 4. Now 2014 is 2 mod 4, so we haven't ruled out 4 being delicious. Consider the 4 integers 1, 2, 3, and 2008. These 4 integers are positive, add to 2014, and have distinct remainders mod 4. So $\boxed{4}$ is indeed delicious.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 6 There are N students in a class. Each possible nonempty group of students selected a positive integer. All of these integers are distinct and add up to 2014. Compute the greatest possible value of N .

Answer: 5

Solution: The number of nonempty groups of N students is $2^N - 1$. So we have $2^N - 1$ distinct positive integers that add up to 2014. The sum of $2^N - 1$ distinct positive integers is at least the sum of the first $2^N - 1$ positive integers, which is

$$\frac{(2^N - 1)2^N}{2} = (2^N - 1)2^{N-1}.$$

So we have the inequality

$$(2^N - 1)2^{N-1} \leq 2014.$$

If $N \geq 6$, then

$$(2^N - 1)2^{N-1} \geq (2^6 - 1)2^5 = 63(32) = 2016 > 2014.$$

So $N \leq 5$.

Let's check that we can achieve $N = 5$. In that case, $2^N - 1 = 31$. Can we find 31 distinct positive integers that add up to 2014? Let's choose 30 of the integers to be 1, 2, 3, ..., 30. Their sum is

$$\frac{30(31)}{2} = 15(31) = 465.$$

So the 31st number should be $2014 - 465$, which is 1549. Because 1549 is distinct from 1 through 30, we have indeed found 31 distinct positive integers that add up to 2014. So $N = \boxed{5}$ is possible.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 7 If x is a real number and k is a nonnegative integer, recall that the binomial coefficient $\binom{x}{k}$ is defined by the formula

$$\binom{x}{k} = \frac{x(x-1)(x-2)\dots(x-k+1)}{k!}.$$

Compute the value of

$$\frac{\binom{1/2}{2014} \cdot 4^{2014}}{\binom{4028}{2014}}.$$

Express your answer as a fraction in simplest form.

Answer: $\frac{-1}{4027}$ or $-\frac{1}{4027}$

Solution: Let $k = 2014$. Then

$$\begin{aligned} \binom{1/2}{k} &= \frac{(1/2)(1/2-1)(1/2-2)\dots(1/2-k+1)}{k!} \\ &= \frac{(1)(1-2)(1-4)\dots(1-2(k-1))}{k! 2^k} \\ &= (-1)^{k-1} \frac{(2-1)(4-1)\dots(2(k-1)-1)}{k! 2^k} \\ &= (-1)^{k-1} \frac{1 \cdot 3 \cdot 5 \dots (2k-3)}{k! 2^k} \\ &= (-1)^{k-1} \frac{2 \cdot 4 \cdot 6 \dots (2k-2)}{2 \cdot 4 \cdot 6 \dots (2k-2)} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2k-3)}{k! 2^k} \\ &= (-1)^{k-1} \frac{(2k-2)!}{(k-1)! 2^{k-1} k! 2^k} \\ &= 2(-1)^{k-1} \frac{(2k-2)!}{(k-1)! k! 4^k}. \end{aligned}$$

Also

$$\binom{2k}{k} = \frac{(2k)!}{k! k!}.$$

Dividing the first equation by the second gives

$$\begin{aligned} \frac{\binom{1/2}{k} \cdot 4^k}{\binom{2k}{k}} &= 2(-1)^{k-1} \frac{\frac{(2k-2)!}{(k-1)! k!}}{\frac{(2k)!}{k! k!}} \\ &= 2(-1)^{k-1} \frac{(2k-2)!}{(2k)!} \cdot \frac{k!}{(k-1)!} \\ &= 2(-1)^{k-1} \frac{k}{(2k-1)(2k)} \\ &= (-1)^{k-1} \frac{1}{2k-1}. \end{aligned}$$

Because $k = 2014$, we have

$$\frac{\binom{1/2}{k} \cdot 4^k}{\binom{2k}{k}} = (-1)^{2014-1} \frac{1}{2(2014)-1} = \boxed{\frac{-1}{4027}}.$$

Problem 8 A triangle has sides of length $\sqrt{13}$, $\sqrt{17}$, and $2\sqrt{5}$. Compute the area of the triangle.

Answer: 7

Solution: Let $a = \sqrt{13}$, $b = \sqrt{17}$, and $c = 2\sqrt{5}$. Heron's formula says that the area K of the triangle is

$$K = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is the semiperimeter $(a+b+c)/2$. Replacing s in Heron's formula, we get

$$K = \frac{1}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}.$$

The first two factors in the square root multiply to

$$(a+b+c)(b+c-a) = (b+c)^2 - a^2 = 2bc + b^2 + c^2 - a^2.$$

The last two factors in the square root multiply to

$$(a+c-b)(a+b-c) = a^2 - (b-c)^2 = 2bc - (b^2 + c^2 - a^2).$$

Hence the product of all four factors is

$$(2bc)^2 - (b^2 + c^2 - a^2)^2 = 4b^2c^2 - (b^2 + c^2 - a^2)^2.$$

So Heron's formula becomes

$$K = \frac{1}{4}\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}.$$

Let's plug in the particular values of a , b , and c . We have $a^2 = 13$, $b^2 = 17$, and $c^2 = 20$. So

$$b^2 + c^2 - a^2 = 17 + 20 - 13 = 24.$$

Hence the area of the triangle is

$$K = \frac{1}{4}\sqrt{4(17)(20) - 24^2} = \sqrt{17(5) - 6^2} = \sqrt{85 - 36} = \sqrt{49} = \boxed{7}.$$

Alternative Solution: Consider the triangle whose vertices are at $(0, 0)$, $(-2, 3)$, and $(2, 4)$. Its sides have the desired lengths $\sqrt{13}$, $\sqrt{17}$, and $2\sqrt{5}$. We can find the area of the triangle from its coordinates in lots of ways. For example, we can apply the shoelace formula or Pick's theorem, or bound the triangle by a 4-by-4 square.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 9 Let abc be a three-digit prime number whose digits satisfy $a < b < c$. The difference between every two of the digits is a prime number too. What is the sum of all the possible values of the three-digit number abc ?

Answer: 736

Solution: The differences $b - a$, $c - b$, and $c - a$ are prime numbers less than 9, that is 2, 3, 5, or 7. Because $c - a$ is the sum of $b - a$ and $c - b$, two of the prime differences add up to a third prime. The only possibilities are $2 + 5$ (or $5 + 2$) and $2 + 3$ (or $3 + 2$).

Assume that $b - a$ is 2 and $c - b$ is 5. Then $c - a$ is 7. So abc is $aaa + 27$, a multiple of 3. But that contradicts abc being prime.

A similar analysis rules out the case when $b - a$ is 5 and $c - b$ is 2.

Assume that $b - a$ is 2 and $c - b$ is 3. The candidates for abc are 136, 247, 358, and 469. But 136 and 358 are even, 247 is a multiple of 13, and 469 is a multiple of 7. So none of these candidates is prime.

Assume that $b - a$ is 3 and $c - b$ is 2. The candidates for abc are 146, 257, 368, and 479. But 146 and 368 are even and hence not prime. On the other hand, 257 and 479 are prime.

In the summary, the possible values of abc are 257 and 479. Their sum is $\boxed{736}$.

Problem 10 An ant is on one face of a cube. At every step, the ant walks to one of its four neighboring faces with equal probability. What is the expected (average) number of steps for it to reach the face opposite its starting face?

Answer: 6

Solution: Let A , B , C , D , E , and F be the faces of the cube, with A the starting face. Suppose that A and F are opposite, B and E are opposite, and C and D are opposite. Let a be the number we are looking for, the expected number of steps from A to F . Let b be the expected number of steps from B to F . By symmetry, b is also the expected number of steps from C (or D or E) to F .

Starting from A , after 1 step, the ant will be at B or C or D or E . So $a = 1 + b$.

From B , after 1 step, the ant will be at A with probability $\frac{1}{4}$, at C or D with combined probability $\frac{1}{2}$, and at F with probability $\frac{1}{4}$. So $b = 1 + \frac{1}{4}a + \frac{1}{2}b$. Simplifying gives $a = 2b - 4$. Because $a = 1 + b$, we get the equation $2b - 4 = 1 + b$. So $b = 5$. Hence $a = 1 + b = 1 + 5 = \boxed{6}$.

Problem 11 Let R be the set of points (x, y) such that $\lfloor x^2 \rfloor = \lfloor y \rfloor$ and $\lfloor y^2 \rfloor = \lfloor x \rfloor$. Compute the area of region R . Express your answer in the form $a - b\sqrt{c}$, where a and b are positive integers and c is a square-free positive integer. Recall that $\lfloor z \rfloor$ is the greatest integer that is less than or equal to z .

Answer: $4 - 2\sqrt{2}$

Solution: Let (x, y) be a point in R . Because squares are nonnegative, we have

$$\lfloor x \rfloor = \lfloor y^2 \rfloor \geq 0.$$

So $x \geq 0$. Similarly, $y \geq 0$.

Let $m = \lfloor x \rfloor$ and $n = \lfloor y \rfloor$. By the previous paragraph, $m \geq 0$ and $n \geq 0$. Because $x \geq m$, we have $x^2 \geq m^2$. So

$$n = \lfloor y \rfloor = \lfloor x^2 \rfloor \geq m^2.$$

Similarly, we have $m \geq n^2$.

Because $n \geq m^2$ and $m \geq n^2$, we have

$$n \geq m^2 \geq (n^2)^2 = n^4.$$

Because n is an integer such that $n \geq n^4$, it follows that n is 0 or 1. Similarly, m is 0 or 1. To make $n \geq m^2$ and $m \geq n^2$ both true, either $m = n = 0$ or $m = n = 1$.

First, let's handle the case $m = n = 0$. In that case, $0 \leq x < 1$ and $0 \leq y < 1$. So $0 \leq x^2 < 1$ and $0 \leq y^2 < 1$. As a result, the equations $\lfloor x^2 \rfloor = \lfloor y \rfloor$ and $\lfloor y^2 \rfloor = \lfloor x \rfloor$ are automatically satisfied. So the entire unit square $[0, 1) \times [0, 1)$ is in R .

Next, let's handle the case $m = n = 1$. Then $\lfloor x^2 \rfloor = \lfloor y \rfloor = n = 1$; so $1 \leq x^2 < 2$, which means that $1 \leq x < \sqrt{2}$. Similarly, $1 \leq y < \sqrt{2}$. So the square $[1, \sqrt{2}) \times [1, \sqrt{2})$ is in R .

Combining the two previous paragraphs, we see that region R is the disjoint union of the squares $[0, 1) \times [0, 1)$ and $[1, \sqrt{2}) \times [1, \sqrt{2})$. The first square has area 1. The second square has area

$$(\sqrt{2} - 1)^2 = 3 - 2\sqrt{2}.$$

So the total area of R is

$$1 + (3 - 2\sqrt{2}) = \boxed{4 - 2\sqrt{2}}.$$

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 12 Let B be a $1 \times 2 \times 4$ box (rectangular parallelepiped). Let R be the set of points that are within distance 3 of some point in B . (Note that R contains B .) What is the volume of R ? Express your answer in terms of π .

Answer: $99\pi + 92$ or $92 + 99\pi$

Solution: Call the dimensions of the box $a = 1$, $b = 2$, and $c = 4$. Let $r = 3$ be the distance threshold.

The volume of B itself is abc .

Consider one of the faces of B , say one of dimension a by b . Then part of R is a box of dimension a by b by r on that face. The volume of that box is abr . By considering each of the 6 faces of B , we get 6 boxes in R with total volume $2r(ab + ac + bc)$.

Consider one of the edges of B , say one of length a . Then part of R is a quarter cylinder of height a and radius r whose axis is that edge. The volume of that quarter cylinder is $\frac{1}{4}r^2a$. By considering each of the 12 edges of B , we get 12 quarter cylinders in R with total volume $\pi r^2(a + b + c)$.

Finally, consider one of the vertices of B . Then part of R is one-eighth of a ball of radius r whose center is that vertex. The volume of that eighth of a ball is $\frac{1}{8} \cdot \frac{4}{3}\pi r^3$. By considering each of the 8 vertices of B , we get 8 eighth balls in R with total volume $\frac{4}{3}\pi r^3$.

The region R is the disjoint union of the parts we have listed in the last four paragraphs. So the volume of R is

$$abc + 2r(ab + ac + bc) + \pi r^2(a + b + c) + \frac{4}{3}\pi r^3.$$

Plugging in $a = 1$, $b = 2$, $c = 4$, and $r = 3$, we find the volume of R to be

$$1(2)(4) + 2(3)(2 + 4 + 8) + \pi(3^2)(1 + 2 + 4) + \frac{4}{3}\pi(3^3) = \boxed{92 + 99\pi}.$$

Problem 13 Deepali has a bag containing 10 red marbles and 10 blue marbles (and nothing else). She removes a random marble from the bag. She keeps doing so until all of the marbles remaining in the bag have the same color. Compute the probability that Deepali ends with exactly 3 marbles remaining in the bag. Express your answer as a fraction in simplest form.

Answer: $\frac{40}{323}$

Solution: We'll first compute the probability that Deepali ends with exactly 3 *red* marbles remaining. We'll then double that probability to account for her possibly ending with 3 blue marbles instead.

Even though Deepali is supposed to stop once all the remaining marbles have the same color, let's imagine that she continues until she has removed all 20 marbles. For Deepali to end with exactly 3 red marbles in the original game, her final four marbles in the extended game have to be BRRR (blue, red, red, red). Let's analyze the probability of finishing with BRRR, starting from the end. The probability that the final marble is red is $\frac{10}{20}$, or $\frac{1}{2}$. Given that the final marble is red, the probability that the second-to-last marble is red is $\frac{9}{19}$. Given that the final two marbles are red, the probability that the third-to-last marble is red is $\frac{8}{18}$, or $\frac{4}{9}$. Given that the final three marbles are red, the probability that the fourth-to-last marble is blue is $\frac{10}{17}$. So the probability that the final four marbles are BRRR is

$$\frac{1}{2} \cdot \frac{9}{19} \cdot \frac{4}{9} \cdot \frac{10}{17} = \frac{20}{19 \cdot 17} = \frac{20}{323}.$$

That's the probability of ending with exactly 3 red marbles in the original game. So the probability of ending with 3 marbles (of either color) in the original game is twice that, or $\boxed{\frac{40}{323}}$.

Problem 14 A triangle has area 114 and sides of integer length. What is the perimeter of the triangle?

Answer: 76

Solution: Let a , b , and c be the side lengths of the triangle. Let s be its semiperimeter $(a + b + c)/2$. By Heron's formula, we get the equation

$$\sqrt{s(s-a)(s-b)(s-c)} = 114.$$

Squaring, we get

$$s(s-a)(s-b)(s-c) = 114^2.$$

Because the side lengths are integers, the semiperimeter s is either an integer or a half integer (half an odd integer). If s were a half integer, then $s-a$, $s-b$, and $s-c$ would be too; but then $s(s-a)(s-b)(s-c)$ couldn't be an integer. So s is an integer.

Let $x = s - a$, $y = s - b$, and $z = s - c$. Their sum is

$$x + y + z = (s - a) + (s - b) + (s - c) = 3s - (a + b + c) = 3s - 2s = s.$$

So our main equation becomes

$$sxyz = 114^2,$$

where x , y , and z add up to s .

To proceed further, we will prime factorize: $114 = 2 \cdot 3 \cdot 19$. So $114^2 = 2^2 \cdot 3^2 \cdot 19^2$. Our main equation becomes

$$sxyz = 2^2 \cdot 3^2 \cdot 19^2.$$

We claim that s is a multiple of 19. Assume not. Then s is a factor of $2^2 \cdot 3^2 = 36$, and in particular is at most 36. In contrast, either two of x , y , and z are multiples of 19 or one of them is a multiple of 19^2 ; either way, their sum is at least 38. But that contradicts s being their sum. So s is a multiple of 19.

We claim that x , y , or z is a multiple of 19. Assume not. Then each of them is a factor of $2^2 \cdot 3^2 = 36$, and in particular each is at most 36. In contrast, s is a multiple of $19^2 = 361$, and so is at least 361. But that contradicts s being the sum of x , y , and z . So x , y , or z is a multiple of 19.

By the two preceding paragraphs, we know that s is a multiple of 19 and so is x , y , or z . Without loss of generality, assume that x is a multiple of 19. Then $y + z = s - x$ is a multiple of 19 too. And yz is a factor of $2^2 \cdot 3^2 = 36$. The only possibility is for y and z to be 1 and 18 (in some order). That means $sx = 2 \cdot 19^2$. So $x = 19$ and $s = 2 \cdot 19 = 38$. Since the semiperimeter s is 38, the perimeter is $\boxed{76}$.

Problem 15 There are two math exams called A and B. 2014 students took the A exam and/or the B exam. Each student took one or both exams, so the total number of exam papers was between 2014 and 4028, inclusive. The score for each exam is an integer from 0 through 40. The average score of all the exam papers was 20. The grade for a student is the best score from one or both exams that she took. The average grade of all 2014 students was 14. Let G be the *greatest* possible number of students who took both exams. Let L be the *least* possible number of students who took both exams. Compute $G - L$.

Answer: 200

Solution: Because the average grade of all 2014 students is 14, the sum of all 2014 student grades is $14(2014)$.

Let n be the number of students who took both exams. Then the number of students who took only one exam is $2014 - n$. So the total number of exam papers is

$$2(n) + 1(2014 - n) = n + 2014.$$

Because the average score of all $n + 2014$ exam papers is 20, the sum of all exam scores is $20(n + 2014)$.

What is the difference between the sum of all exam scores and the sum of all student grades? The first sum includes the lower score for each of the n students who took both exams. Because each score is at most 40, the first sum is at most $40n$ more than the second sum. By our previous work, we get the inequality

$$20(n + 2014) \leq 14(2014) + 40n.$$

Solving for n gives

$$n \geq \frac{6(2014)}{20} = \frac{3(2014)}{10} = 604.2.$$

So $n \geq 605$.

Let's check that $n = 605$ is achievable. The inequalities in the previous paragraph are nearly sharp when the lower scores for the n students who took both exams are nearly 40. With that in mind, here is one scenario for achieving 605. Suppose that 604 students score 40 and 40, 1 student scores 25 and 24, 1337 (leet!) students score 3, and 72 students score 0. So L , the least possible value of n , is 605.

We will now compute an upper bound on n . Again we will compare the sum of all exam scores and the sum of all student grades. We claim that the first sum is at most double the second sum. That's because for each student who took both exams, the first sum includes her lower and upper score, whereas double the second sum includes her upper score twice; also, for each student who took only one exam, the first sum includes her score once, whereas double the second sum includes her score twice. By our work in the first two paragraphs, we get the inequality

$$20(n + 2014) \leq 2 \cdot 14(2014).$$

Solving for n gives

$$n \leq \frac{8(2014)}{20} = \frac{4(2014)}{10} = 805.6.$$

So $n \leq 805$.

Let's show that $n = 805$ is achievable. The inequalities in the previous paragraph are nearly sharp when the lower and upper scores for the n students who took both exams are nearly equal, and the scores for the 2014 – n students who took only one exam are nearly 0. With that in mind, here is one scenario for achieving 805. Suppose that 802 students score 35 and 35, 3 students score 38 and 38, 1208 students score 0, and 1 student scores 12. So G , the greatest possible value of n , is 805.

Hence $G - L$, the desired quantity, is $805 - 605 = \boxed{200}$.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 16 If $\sin x + \sin y = \frac{96}{65}$ and $\cos x + \cos y = \frac{72}{65}$, then what is the value of $\tan x + \tan y$? Express your answer as a fraction in simplest form.

Answer: $\frac{507}{112}$

Solution: Recall the sum-to-product identities

$$\begin{aligned}\sin x + \sin y &= 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} \\ \cos x + \cos y &= 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}.\end{aligned}$$

So our given equations become

$$\begin{aligned}2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} &= \frac{96}{65} \\ 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2} &= \frac{72}{65}.\end{aligned}$$

To simplify these equations, let $a = \frac{x+y}{2}$ and $b = \frac{x-y}{2}$. Then the equations become

$$\begin{aligned}2 \sin a \cos b &= \frac{96}{65} \\ 2 \cos a \cos b &= \frac{72}{65}.\end{aligned}$$

Dividing by 2 gives

$$\begin{aligned}\sin a \cos b &= \frac{48}{65} \\ \cos a \cos b &= \frac{36}{65}.\end{aligned}$$

In this last pair of equations, if we divide the first equation by the second, we get

$$\tan a = \frac{48}{36} = \frac{4}{3}.$$

On the other hand, if we square both equations and add, we get

$$\cos^2 b = \left(\frac{48}{65}\right)^2 + \left(\frac{36}{65}\right)^2 = \left(\frac{12}{13}\right)^2.$$

Hence we have

$$\tan^2 b = \sec^2 b - 1 = \left(\frac{13}{12}\right)^2 - 1 = \left(\frac{5}{12}\right)^2.$$

So $\tan b = \pm \frac{5}{12}$.

At first, we will assume that $\tan b = \frac{5}{12}$. Later we will handle the opposite case.

Because $a = \frac{x+y}{2}$ and $b = \frac{x-y}{2}$, we have $x = a + b$ and $y = a - b$. By the tangent sum formula, we have

$$\tan x = \tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} = \frac{4/3 + 5/12}{1 - (4/3)(5/12)} = \frac{63}{16}.$$

Similarly, by the tangent difference formula, we have

$$\tan y = \tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} = \frac{4/3 - 5/12}{1 + (4/3)(5/12)} = \frac{33}{56}.$$

The previous paragraph assumed that $\tan b = \frac{5}{12}$. If $\tan b$ were $-\frac{5}{12}$, then the calculation would be similar, except that the values of $\tan x$ and $\tan y$ would be switched. Either way, their sum is

$$\tan x + \tan y = \frac{63}{16} + \frac{33}{56} = \frac{441}{112} + \frac{66}{112} = \boxed{\frac{507}{112}}.$$

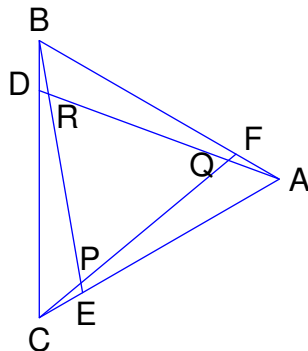
Problem 17 Let ABC be a triangle. Points D , E , and F are respectively on the sides \overline{BC} , \overline{CA} , and \overline{AB} of $\triangle ABC$. Suppose that

$$\frac{AE}{AC} = \frac{CD}{CB} = \frac{BF}{BA} = x$$

for some x with $\frac{1}{2} < x < 1$. Segments \overline{AD} , \overline{BE} , and \overline{CF} cut the triangle into 7 nonoverlapping regions: 4 triangles and 3 quadrilaterals. The total area of the 4 triangles equals the total area of the 3 quadrilaterals. Compute the value of x . Express your answer in the form $\frac{k-\sqrt{m}}{n}$, where k and n are positive integers and m is a square-free positive integer.

Answer: $\frac{11 - \sqrt{37}}{6}$

Solution: Below is a diagram of the situation.



Let $r = \frac{BF}{FA}$. Because $\frac{BF}{BA} = x$, we have $r = \frac{x}{1-x}$. Because $\frac{1}{2} < x < 1$, we have $r > 1$. The barycentric coordinates of F are $r : 1 : 0$. Similarly, $D = 0 : r : 1$ and $E = 1 : 0 : r$.

Let P be the intersection point of \overline{BE} and \overline{CF} , let Q be the intersection point of \overline{AD} and \overline{CF} , and let R be the intersection point of \overline{AD} and \overline{BE} . Because P lies on \overline{BE} , the ratio of its C -coordinate to A -coordinate is r . Because P lies on \overline{CF} , the ratio of its A -coordinate to B -coordinate is also r . So the barycentric coordinates of P are $r : 1 : r^2$. Similarly, $Q = r^2 : r : 1$ and $R = 1 : r^2 : r$.

By scaling, we may assume that the area of $\triangle ABC$ is 1. Recall that barycentric coordinates are also areal coordinates. Because $D = 0 : r : 1$, the area of $\triangle ABD$ is $\frac{1}{r+1}$. Similarly, $\triangle BCE$ and $\triangle CAF$ each have the same area $\frac{1}{r+1}$.

Because $P = r : 1 : r^2$, the area of $\triangle BCP$ is $\frac{r}{r^2+r+1}$. Similarly, $\triangle AQC$ and $\triangle ABR$ each have the same area $\frac{r}{r^2+r+1}$.

Because the area of $\triangle CAF$ is $\frac{1}{1+r}$ and the area of $\triangle AQC$ is $\frac{r}{r^2+r+1}$, the area of $\triangle AQF$ is their difference

$$\frac{1}{1+r} - \frac{r}{r^2+r+1} = \frac{1}{(r+1)(r^2+r+1)}.$$

Similarly, $\triangle BDR$ and $\triangle CEP$ each have the same area.

Quadrilateral $CPRD$ is $\triangle BCE$ minus $\triangle DBR$ and $\triangle CPE$, so the area of $CPRD$ is

$$\frac{1}{r+1} - \frac{2}{(r+1)(r^2+r+1)} = \frac{r^2+r-1}{(r+1)(r^2+r+1)}.$$

Similarly, the quadrilaterals $BRQF$ and $AQPE$ each have the same area.

The problem says that the total area of the three quadrilaterals is half the area of $\triangle ABC$. So we get the equation

$$3 \frac{r^2+r-1}{(r+1)(r^2+r+1)} = \frac{1}{2}.$$

Clearing fractions and bringing all terms to one side gives

$$r^3 - 4r^2 - 4r + 7 = 0.$$

Factoring gives

$$(r-1)(r^2-3r-7) = 0.$$

Because $r > 1$, we have

$$r^2 - 3r - 7 = 0.$$

By the quadratic formula,

$$r = \frac{3 \pm \sqrt{37}}{2}.$$

Because $r > 1$, we have

$$r = \frac{\sqrt{37} + 3}{2}.$$

Recall that $r = \frac{x}{1-x}$. Solving for x gives $x = 1 - \frac{1}{r+1}$. Hence

$$x = 1 - \frac{2}{\sqrt{37} + 5} = 1 - \frac{2(\sqrt{37} - 5)}{37 - 5^2} = \boxed{\frac{11 - \sqrt{37}}{6}}.$$

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 18 For how many integers k such that $0 \leq k \leq 2014$ is it true that the binomial coefficient $\binom{2014}{k}$ is a multiple of 4?

Answer: 991

Solution: We will count the opposite: the number of integers k (with $0 \leq k \leq 2014$) such that $\binom{2014}{k}$ is *not* a multiple of 4. An integer that is not a multiple of 4 either is odd (has no factors of 2) or has exactly one factor of 2.

By Kummer's theorem, the number of factors of 2 in $\binom{2014}{k}$ is the number of carries when k and $2014 - k$ are added in base 2 (binary). So $\binom{2014}{k}$ not being a multiple of 4 means that when k and $2014 - k$ are added in binary, the number of carries is either 0 or 1.

Let's compute the binary representation of 2014. Because 2014 is just a little less than a power of 2 (namely 34 less than $2048 = 2^{11}$), it's easy to find the binary representation of 2014. Namely, $2014 = 11111011110_2$. We will number the bit positions 0, 1, 2, \dots , starting from the right. So 2014 has 0's in positions 0 and 5, and has 1's in the other 9 positions.

Let's first count the number of k such that $\binom{2014}{k}$ is odd. In that case, there won't be any carries when k and $2014 - k$ are added in binary. For that to work out, k must have 0's in positions 0 and 5 (matching 2014), but the other 9 bits of k can be arbitrary. So the number of such k is $2^9 = 512$.

Next, let's count the number of k such that $\binom{2014}{k}$ has exactly one factor of 2 (a so-called "oddly even" or "singly even" number). In that case, there will be exactly one carry when k and $2014 - k$ in binary. There are two possibilities: k either has 0's in positions 1 and 5 and a 1 in position 0 (subcase 1) or has 0's in positions 0 and 6 and a 1 in position 5 (subcase 2). In subcase 1, the value of k has 3 fixed bits and 8 arbitrary bits, so the number of such k is $2^8 = 256$. In subcase 2, the value of k also has 3 fixed bits and 8 arbitrary bits, so the number of such k is again $2^8 = 256$. Hence the number of k such that $\binom{2014}{k}$ has exactly one factor of 2 is $256 + 256 = 512$.

Combining the two previous paragraphs, we see that the number of integers k (with $0 \leq k \leq 2014$) such that $\binom{2014}{k}$ is not a multiple of 4 is $512 + 512 = 1024$. So the number of k such that $\binom{2014}{k}$ is a multiple of 4 is $2015 - 1024 = \boxed{991}$.

Problem 19 Let n be a positive integer. Let (a, b, c) be a random ordered triple of nonnegative integers such that $a + b + c = n$, chosen uniformly at random from among all such triples. Let M_n be the expected value (average value) of the largest of a , b , and c . As n approaches infinity, what value does

$\frac{M_n}{n}$ approach? Express your answer as a fraction in simplest form.

Answer: $\frac{11}{18}$

Solution:

Call a triple (a, b, c) *valid* if $a, b,$ and c are nonnegative integers such that $a + b + c = n$. We will first count the number of valid triples. By the “stars and bars” method, we can view this count as the number of ways to arrange n identical objects (“stars”) and 2 identical dividers (“bars”). Out of $n + 2$ possible positions, we have to choose 2 positions for the dividers. So the count is

$$\binom{n+2}{2} = \frac{(n+2)(n+1)}{2} = \frac{1}{2}n^2 + O(n).$$

Let m be an integer. We will count the number of valid triples (a, b, c) such that $\max(a, b, c) = m$. If (a, b, c) is such a triple, then

$$m = \max(a, b, c) \leq a + b + c = n.$$

In the opposite direction, we have

$$m = \max(a, b, c) \geq \frac{a + b + c}{3} = \frac{n}{3}.$$

So we may assume that $\frac{n}{3} \leq m \leq n$. If $m = \frac{n}{3}$, then the only possible such triple is (m, m, m) , which we can ignore as negligible. So we will assume that $\frac{n}{3} < m \leq n$.

To count the number of valid triples (a, b, c) such that $\max(a, b, c) = m$, we will divide into two cases, depending on the value of m .

Case $\frac{n}{3} < m \leq \frac{n}{2}$: The only valid triples (a, b, c) in which the maximum m is achieved by two of the variables are the three triples $(m, m, n - 2m)$, $(m, n - 2m, m)$, and $(n - 2m, m, m)$. Otherwise, the maximum is achieved by exactly one of the variables. Let’s temporarily assume that the maximum is achieved uniquely by c , and later multiply the count by 3 to account for the maximum being $a, b,$ or c . So $c = m, a < m,$ and $b < m$. Because $c = m$, we have $a + b = n - c = n - m$. So the only pairs (a, b) are

$$(n - 2m + 1, m - 1), (n - 2m + 2, m - 2), \dots, (m - 2, n - 2m + 2), (m - 1, n - 2m + 1).$$

The number of such pairs is

$$(m - 1) - (n - 2m + 1) + 1 = 3m - n - 1.$$

So the number of valid triples (a, b, c) such that the maximum m is achieved uniquely is three times this count, or $9m - 3n - 3$. Hence the number of valid triples (a, b, c) such that $\max(a, b, c) = m$ is 3 more than this count, or $9m - 3n$.

Case $\frac{n}{2} < m \leq n$: Because $m > \frac{n}{2}$, the maximum $\max(a, b, c) = m$ is achieved uniquely. Assume for the moment that c is the maximum m . Then $a + b = n - c = n - m$. So the only pairs (a, b) are

$$(0, n - m), (1, n - m - 1), \dots, (n - m - 1, 1), (n - m, 0).$$

The number of such pairs is $n - m + 1$. Hence the number of valid triples (a, b, c) such that $\max(a, b, c) = m$ is three times this count, or $3n - 3m + 3$.

We can sum up $\max(a, b, c)$ over all valid triples (a, b, c) . We have

$$\begin{aligned} \sum_{a,b,c} \max(a, b, c) &= \sum_m \sum_{\substack{a,b,c \\ \max(a,b,c)=m}} \max(a, b, c) \\ &= \sum_m \sum_{\substack{a,b,c \\ \max(a,b,c)=m}} m \\ &= \sum_m m |\{(a, b, c) : \max(a, b, c) = m\}|. \end{aligned}$$

As before, we will split the possible m into two cases. (The third case $m = \frac{n}{3}$ contributes at most $O(n)$ to the sum.) We have

$$\begin{aligned} \sum_{a,b,c} \max(a, b, c) &= O(n) + \sum_{\substack{m \\ n/3 < m \leq n/2}} m(9m - 3n) + \sum_{\substack{m \\ n/2 < m \leq n}} m(3n - 3m + 3) \\ &= O(n) + 9 \sum_{\substack{m \\ n/3 < m \leq n/2}} m^2 - 3n \sum_{\substack{m \\ n/3 < m \leq n/2}} m \\ &\quad + (3n + 3) \sum_{\substack{m \\ n/2 < m \leq n}} m - 3 \sum_{\substack{m \\ n/2 < m \leq n}} m^2. \end{aligned}$$

Recall that the sum of the first k positive integers is

$$\frac{k(k+1)}{2} = \frac{1}{2}k^2 + O(k).$$

Similarly, the sum of the squares of the first k positive integers is

$$\frac{k(k+1)(2k+1)}{6} = \frac{1}{3}k^3 + O(k^2).$$

So we can estimate our previous sums by

$$\begin{aligned}
 \sum_{a,b,c} \max(a, b, c) &= O(n^2) + 9\left[\frac{1}{3}\binom{n}{2}^3 - \frac{1}{3}\binom{n}{3}^3\right] - 3n\left[\frac{1}{2}\binom{n}{2}^2 - \frac{1}{2}\binom{n}{3}^2\right] \\
 &\quad + (3n+3)\left[\frac{1}{2}n^2 - \frac{1}{2}\binom{n}{2}^2\right] - 3\left[\frac{1}{3}n^3 - \frac{1}{3}\binom{n}{2}^3\right] \\
 &= O(n^2) + 9\left[\frac{1}{3} \cdot \frac{19}{216}n^3\right] - 3n\left[\frac{1}{2} \cdot \frac{5}{36}n^2\right] \\
 &\quad + (3n+3)\left[\frac{1}{2} \cdot \frac{3}{4}n^2\right] - 3\left[\frac{1}{3} \cdot \frac{7}{8}n^3\right] \\
 &= O(n^2) + \frac{19}{72}n^3 - \frac{15}{72}n^3 + \frac{9}{8}n^3 - \frac{7}{8}n^3 \\
 &= \frac{11}{36}n^3 + O(n^2).
 \end{aligned}$$

Finally, we can estimate M_n , the expected value of $\max(a, b, c)$ over all valid triples. Recall that the number of valid triples is $\frac{1}{2}n^2 + O(n)$. So M_n is

$$M_n = \frac{\frac{11}{36}n^3 + O(n^2)}{\frac{1}{2}n^2 + O(n)} = \frac{11}{18}n + O(1).$$

Dividing by n gives

$$\frac{M_n}{n} = \frac{11}{18} + O\left(\frac{1}{n}\right).$$

So as n approaches infinity, $\frac{M_n}{n}$ approaches $\boxed{\frac{11}{18}}$.

Problem 20 How many complex numbers z such that $|z| < 30$ satisfy the equation

$$e^z = \frac{z-1}{z+1}?$$

Answer: 10

Solution: Let $z = x + yi$, where x and y are real. Then

$$|e^z| = |e^{x+yi}| = |e^x \cdot e^{iy}| = |e^x| \cdot |e^{iy}| = e^x \cdot 1 = e^x.$$

So e^z is inside the unit circle if $x < 0$, is on the unit circle if $x = 0$, and is outside the unit circle if $x > 0$.

Also, note that z is closer to -1 than to 1 if $x < 0$, is equidistant to 1 and -1 if $x = 0$, and is closer to 1 than to -1 if $x > 0$. So $\frac{z-1}{z+1}$ is outside the unit circle (or undefined) if $x < 0$, is on the unit circle if $x = 0$, and is inside the unit circle if $x > 0$.

Comparing the two previous paragraphs, we see that if $e^z = \frac{z-1}{z+1}$, then $x = 0$. So z is the purely imaginary number yi .

Also, note that z satisfies the original equation if and only if $-z$ does. So at first we will assume that y is positive, and at the end we will double the number of roots to account for negative y . (Note that $y \neq 0$, because $z = 0$ is not a root of the original equation.)

Substituting $z = yi$ into the equation $e^z = \frac{z-1}{z+1}$ gives the new equation

$$e^{iy} = \frac{iy - 1}{iy + 1}.$$

By the first two paragraphs, we know that both sides of the equation are always on the unit circle. The only thing we don't know is when the two sides are at the same point on the unit circle.

Given a nonzero complex number w , the *angle* of w (often called the argument of w) is the angle in the interval $[0, 2\pi)$ that the segment from 0 to w makes with the positive x -axis. (In other words, the angle when w is written in polar form.)

Let's reason about angles. As y increases from 0 to ∞ , the angle of $iy - 1$ strictly decreases from π to $\frac{\pi}{2}$, while the angle of $iy + 1$ strictly increases from 0 to $\frac{\pi}{2}$. So the angle of $\frac{iy-1}{iy+1}$ strictly decreases from π to 0 .

Let n be a nonnegative integer. We will consider y in the interval from $2n\pi$ to $(2n + 2)\pi$.

As y increases from $2n\pi$ to $(2n + 1)\pi$, the angle of e^{iy} strictly increases from 0 to π . As y increases from $(2n + 1)\pi$ to just under $(2n + 2)\pi$, the angle of e^{iy} strictly increases from π to just under 2π .

Comparing the angle information for $\frac{iy-1}{iy+1}$ and e^{iy} above, we see that $\frac{iy-1}{iy+1}$ and e^{iy} are equal for exactly one y in $(2n\pi, (2n + 1)\pi)$, and for no y in $[(2n + 1)\pi, (2n + 2)\pi]$. So we have exactly one root of y in each of $(0, \pi)$, $(2\pi, 3\pi)$, $(4\pi, 5\pi)$, $(6\pi, 7\pi)$, and $(8\pi, 9\pi)$. That gives 5 positive roots for y . We don't have to go further because $9\pi < 30 < 10\pi$.

Because we have 5 positive roots for y , by symmetry we have 5 negative roots for y . Altogether, the total number of roots is $\boxed{10}$.