The Advantage Testing Foundation Math Prize for Girls Olympiad

Thursday, November 12, 2015

Time Limit: 4 hours

1. Prove that every positive integer has a unique representation in the form

$$\sum_{i=0}^{k} d_i 2^i \,,$$

where k is a nonnegative integer and each d_i is either 1 or 2. (This representation is similar to usual binary notation except that the digits are 1 and 2, not 0 and 1.)

2. A tetrahedron T is inside a cube C. Prove that the volume of T is at most one-third the volume of C.

3. Let f be the cubic polynomial

$$f(x) = x^3 + bx^2 + cx + d,$$

where b, c, and d are real numbers. Let x_1, x_2, $\ldots$, x_n be nonnegative numbers, and let m be their average. Suppose that $m \geq -\dfrac{b}{2}$. Prove that

$$\sum_{i=1}^{n} f(x_i) \geq n f(m).$$

4. An 8-by-8 square is divided into 64 unit squares in the usual way. Each unit square is colored black or white. The number of black unit squares is even. We can take two adjacent unit squares (forming a 1-by-2 or 2-by-1 rectangle), and flip their colors: black becomes white and white becomes black. We call this operation a *step*. If C is the original coloring, let $S(C)$ be the least number of steps required to make all the unit squares black. Find with proof the greatest possible value of $S(C)$.