The Advantage Testing Foundation

Math Prize for Girls at MIT

2015 Olympiad Solutions

Problem 1 Prove that every positive integer has a unique representation in the form

$$\sum_{i=0}^{k} d_i 2^i \,,$$

where k is a nonnegative integer and each d_i is either 1 or 2. (This representation is similar to usual binary notation except that the digits are 1 and 2, not 0 and 1.)

Solution: We will call the desired representation a 1-2 representation.

Let n be an integer greater than 2. Let c be 1 if n is odd and 2 if n is even. Then $\frac{n-c}{2}$ is a positive integer. Given a 1-2 representation of $\frac{n-c}{2}$, we will construct a 1-2 representation of n. Namely, suppose the 1-2 representation of $\frac{n-c}{2}$ is $\sum_{i=0}^{k} d_i 2^i$. Then

$$n = c + 2\sum_{i=0}^{k} d_i 2^i = c + \sum_{i=0}^{k} d_i 2^{i+1} = c \cdot 2^0 + \sum_{i=1}^{k+1} d_{i-1} 2^i .$$

The final expression is a 1-2 representation of n. So we have a mapping from the set of 1-2 representations of $\frac{n-c}{2}$ to the set of 1-2 representations of n.

Similarly, given a 1-2 representation of n, we can construct a 1-2 representation of $\frac{n-c}{2}$. Namely, suppose the 1-2 representation of n is $\sum_{i=0}^{k} d_i 2^i$. Looking mod 2, we see that n and d_0 must have the same parity. In particular, d_0 must be c. Therefore

$$\frac{n-c}{2} = \frac{1}{2}\sum_{i=1}^{k} d_i 2^i = \sum_{i=1}^{k} d_i 2^{i-1} = \sum_{i=0}^{k-1} d_{i+1} 2^i .$$

The final expression is a 1-2 representation of $\frac{n-c}{2}$. So we have a mapping from the set of 1-2 representations of n to the set of 1-2 representations of $\frac{n-c}{2}$.

We can check that this mapping is the inverse of the mapping in the previous paragraph. So we have a bijection between the set of 1-2 representations of $\frac{n-c}{2}$ and the set of 1-2 representations of n. In particular, the number of 1-2 representations of $\frac{n-c}{2}$ is equal to the number of 1-2 representations of n.

We are now ready to prove that every positive integer has a unique 1-2 representation. Namely, we will show that every positive integer n has exactly one 1-2 representation. The proof will be by strong induction on n. The base cases $n = 1$ and $n = 2$ are trivial. So assume that $n > 2$. As above, let c be 1 if n is odd and 2 if n is even. By induction, $\frac{n-c}{2}$ has exactly one 1-2 representation. By the last sentence in the previous paragraph, n also has exactly one 1-2 representation. So we have proved the induction hypothesis.

Sketch of Alternative Solution: Every 1-2 representation of n corresponds to a binary representation of $n + 1$. Namely, given the 1-2 representation of n, subtract 1 from each digit, and then insert a leading 1; the result is a binary representation of $n+1$. Because $n+1$ has a unique binary representation, it follows that n has a unique 1-2 representation.

Problem 2 A tetrahedron T is inside a cube C. Prove that the volume of T is at most one-third the volume of C.

Solution: We claim that without loss of generality, the vertices of the tetrahedron T are all vertices of the cube C. Let the vertices of T be W, X, Y, and Z. Consider the plane P formed by X, Y, and Z. The volume of T is one-third the area of base triangle XYZ times the distance from W to P. Consider the plane Q parallel to P and passing through W. The plane Q cuts the cube C into two parts, D and E; say that D is the part that doesn't intersect plane P. The part D contains at least one vertex (call it V) of cube C, because a cube is the convex hull of its vertices. The distance from V to P is at least the distance from W to P. So the tetrahedron $VXYZ$ has volume at least that of our original tetrahedron $WXYZ$. By applying the same procedure to X, Y, and Z, we will find a tetrahedron in cube C whose vertices are all vertices of C and whose volume is at least the volume of our original tetrahedron. So if we can show that the problem is true for tetrahedra whose vertices are all vertices of C, then we will be done.

Hence assume that T is a tetrahedron whose vertices are all vertices of the cube C. Without loss of generality, assume that each of the edges of C has length 1.

First, suppose that one face of C contains three vertices of T. Then the area of the base triangle formed by these three vertices is $1/2$, and the altitude from the fourth vertex is 1. Hence the volume of T is $1/6$, which is less than one-third the volume of C.

Next, suppose that every face of C has at most two vertices of T. Then the vertices of T must be four non-adjacent vertices of C. The distance between every pair of vertices of T is $\sqrt{2}$. In other words, T is a regular tetrahedron. The volume of a regular tetrahedron with edge length x is

$$\frac{x^3\sqrt{2}}{12}\,.$$

So the volume of T is

$$\frac{\sqrt{2}^3\sqrt{2}}{12} = \frac{4}{12} = \frac{1}{3}\,.$$

So the volume of T is one-third the volume of C. We have exhausted all cases.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 3 Let f be the cubic polynomial

$$f(x) = x^3 + bx^2 + cx + d,$$

where b, c, and d are real numbers. Let x_1, x_2, …, x_n be nonnegative numbers, and let m be their average. Suppose that $m \geq -\dfrac{b}{2}$. Prove that

$$\sum_{i=1}^{n} f(x_i) \geq nf(m).$$

Solution: Let $y_i = x_i - m$. Note that $y_i \geq -m$ and that the sum of the y_i is zero.

Let g be the function (a translation of f) defined by

$$g(y) = f(y+m) - f(m).$$

We can calculate g as follows:

$$\begin{aligned} g(y) &= f(y+m) - f(m) \\ &= (y+m)^3 + b(y+m)^2 + c(y+m) + d - (m^3 + bm^2 + cm + d) \\ &= y^3 + (3m+b)y^2 + (3m^2 + 2mb + c)y \\ &= (y + 3m + b)y^2 + (3m^2 + 2mb + c)y. \end{aligned}$$

Note that if $y \geq -m$, then

$$y + 3m + b \geq -m + 3m + b = 2m + b \geq -b + b = 0.$$

In particular, if $y \geq -m$, then

$$g(y) \geq (3m^2 + 2mb + c)y.$$

Finally, we can prove the desired inequality:

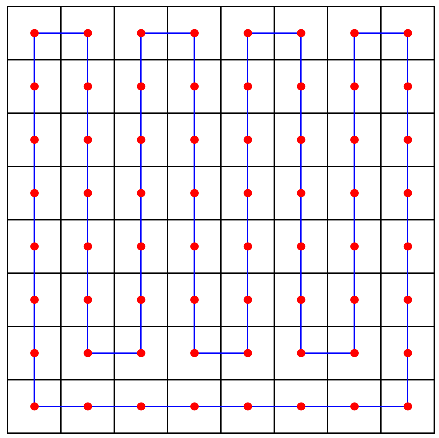
$$\begin{aligned}\sum_{i=1}^{n} f(x_i) &= \sum_{i=1}^{n}[g(y_i) + f(m)] \\ &= nf(m) + \sum_{i=1}^{n} g(y_i) \\ &\geq nf(m) + \sum_{i=1}^{n}(3m^2 + 2mb + c)y_i \\ &= nf(m).\end{aligned}$$

Problem 4 An 8-by-8 square is divided into 64 unit squares in the usual way. Each unit square is colored black or white. The number of black unit squares is even. We can take two adjacent unit squares (forming a 1-by-2 or 2-by-1 rectangle), and flip their colors: black becomes white and white becomes black. We call this operation a *step*. If C is the original coloring, let $S(C)$ be the least number of steps required to make all the unit squares black. Find with proof the greatest possible value of $S(C)$.

Solution: Consider the all-white coloring C. Starting from that coloring, each of the 64 unit squares must change from white to black. Each step involves two unit squares. So the number of steps is at least $64/2 = 32$. Hence $S(C)$ is at least 32.

Now let C be an arbitrary coloring (with an even number of black unit squares). We will show that $S(C)$ is at most 32.

Below is a Hamiltonian cycle on the 8-by-8 square.



Imagine starting at some unit square and walking along this cycle. Let W_0, W_1, …, W_{k-1} be the white unit squares that we see on our walk. (For convenience, set W_k equal to W_0.) Here k is even, because the number of black unit squares (and hence white unit squares) is even. Let d_i be the distance (number of edges) from W_i to W_{i+1} along the walk. Let A be the sum of the even distances $d_0 + d_2 + \cdots + d_{k-2}$. Let B be the sum of the odd distances $d_1 + d_3 + \cdots + d_{k-1}$. The sum of A and B is 64, the length of the entire cycle. In particular, A or B is at most 32.

We will now show that $S(C)$ is at most 32. Perform a flip step at each of the d_0 edges from W_0 to W_1. That sequence of flips will make W_0 and W_1 black, while keeping all the unit squares in between black. Perform a similar sequence of flips from W_2 to W_3, from W_4 to W_5, …, and from W_{k-2} to W_{k-1}. This entire sequence of flips will make all the unit squares black. The number of flip steps is exactly A. So $S(C)$ is at most A. A similar argument shows that $S(C)$ is at most B. So $S(C)$ is at most the minimum of A and B, which is at most 32. Hence $S(C)$ is at most 32.

Combining our lower bound and our upper bound shows that the greatest possible value of $S(C)$ is exactly 32.

Sketch of Alternative Solution: Given two adjacent unit squares in a 2-by-2 square, we can make them both black in at most one step. So in our 8-by-8 square, we can blacken the first row in at most 4 steps. We can do the same for the second row, the third row, the fourth row, the fifth row, and the sixth row, for a total of at most $6 \times 4 = 24$ steps. What remains is a 2-by-8 square. In that 2-by-8 square, we can blacken the first column, the second column, the third column, the fourth column, the fifth column, the sixth column, and the seventh column, in at most 7 more steps. What remains is a 2-by-1 square. Because the number of white squares was and is even, we need at most 1 more step to make every unit square black. The total number of steps is at most $24 + 7 + 1$, or 32.

Note: This problem was proposed by Oleg Kryzhanovsky.