The Advantage Testing Foundation

Math Prize for Girls at MIT

2015 Solutions

Problem 1 In how many different ways can 900 be expressed as the product of two (possibly equal) positive integers? Regard $m \cdot n$ and $n \cdot m$ as the same product.

Answer: 14

Solution: The prime factorization of 900 is $2^2 \cdot 3^2 \cdot 5^2$. So the number of positive divisors of 900 is $3 \cdot 3 \cdot 3 = 27$. These 27 divisors split into 13 pairs each of whose products is 900, plus the solitary divisor 30. So the number of desired pairs is $13 + 1 = \boxed{14}$.

Problem 2 Let x and y be real numbers such that

$$2 < \frac{x-y}{x+y} < 5.$$

If $\frac{x}{y}$ is an integer, what is its value?

Answer: -2

Solution: Let $r = \frac{x}{y}$. Then $\frac{x-y}{x+y} = \frac{r-1}{r+1}$. So our inequality becomes

$$2 < \frac{r-1}{r+1} < 5.$$

Subtracting 1 from all three expressions gives

$$1 < -\frac{2}{r+1} < 4.$$

All three expressions are positive, so we can invert to get

$$\frac{1}{4} < -\frac{r+1}{2} < 1.$$

Multiplying by -2 gives

$$-2 < r + 1 < -\frac{1}{2}\,.$$

Subtracting 1 gives

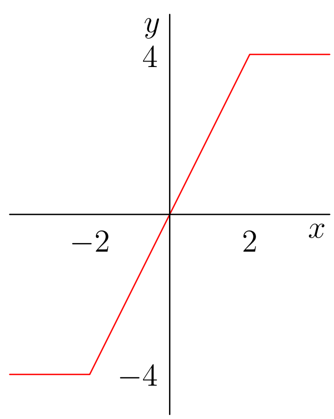
$$-3 < r < -\frac{3}{2}\,.$$

Because r is an integer, r must be $\boxed{-2}$.

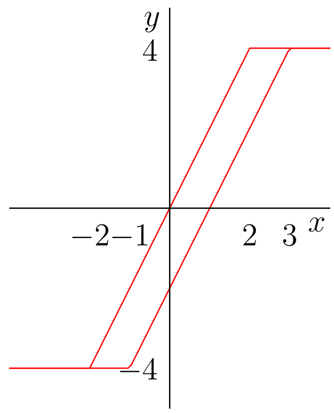
Problem 3 What is the area of the region bounded by the graphs of $y = |x+2| - |x-2|$ and $y = |x+1| - |x-3|$?

Answer: 8

Solution: First let's consider the equation $y = |x+2| - |x-2|$. When $x \le -2$, the value of y is -4. When $x \le 2$, the value of y is 4. And when $-2 \le x \le 2$, the value of y is a linear interpolation between $x = -2$ and $x = 2$. Below is the graph of the equation.



Next, let's consider the equation $y = |x+1| - |x-3|$. Its graph is the graph of $y = |x+2| - |x-2|$ shifted 1 unit to the right. Below is a diagram showing both graphs.



As we can see, the region bounded by the two graphs is a parallelogram. The parallelogram has a base of 1 and a height of 8. So its area is $\boxed{8}$.

Problem 4 A *binary palindrome* is a positive integer whose standard base 2 (binary) representation is a palindrome (reads the same backward or forward). (Leading zeroes are not permitted in the standard representation.) For example, 2015 is a binary palindrome, because in base 2 it is 11111011111. How many positive integers less than 2015 are binary palindromes?

Answer: 92

Solution: Let's first count the number of binary palindromes with between 2 and 11 bits (inclusive). Each such palindrome consists of a first half with between 1 and 5 bits, a possible middle bit, and a second half that is the reverse of the first half. The first half will be one of the binary numbers 1, 10, 11, ..., 11111, the binary representations of the integers 1 through 31; there are 31 such choices. There are 3 choices for the possible middle bit: 0, 1, or nothing. The second half is determined by the first half. So the total number of such binary palindromes is $3 \cdot 31 = 93$.

Next, let's adjust the count to solve the problem. We should include the 1-bit palindrome 1. We should exclude the 11-bit palindromes 11111011111 and 11111111111, because they are 2015 or more. So the final count is $93 + 1 - 2 = \boxed{92}$.

Problem 5 How many distinct positive integers can be expressed in the form $ABCD - DCBA$, where $ABCD$ and $DCBA$ are 4-digit positive integers? (Here A, B, C and D are digits, possibly equal.) Note: A and D can't be zero, because otherwise $ABCD$ or $DCBA$ wouldn't be a true 4-digit integer.

Answer: 161

Solution: In base 10, the number $ABCD$ is $1000A + 100B + 10C + D$. Similarly, $DCBA$ is $1000D + 100C + 10B + A$. So their difference $ABCD - DCBA$ is $999(A-D)+90(B-C)$, which is 9 times $111(A-D)+10(B-C)$. Because A and D are nonzero, $A - D$ is an integer from -8 through 8. Similarly, $B - C$ is an integer from -9 through 9. That gives $17 \cdot 19 = 323$ choices. All 323 choices lead to a distinct difference. Of those 323 differences, one is 0, and the other 322 split evenly into positive and negative. So the number of positive differences is $\boxed{161}$.

Problem 6 In baseball, a player's *batting average* is the number of hits divided by the number of at bats, rounded to three decimal places. Danielle's batting average is .399. What is the fewest number of at bats that Danielle could have?

Answer: 138

Solution: Let H be the number of hits. Let A be the number of at bats. Then $0.3985 \le \frac{H}{A} < 0.3995$. So $0.0005 < \frac{2}{5} - \frac{H}{A} \le 0.0015$. Rewriting gives $0.0005 < \frac{2A-5H}{5A} \le 0.0015$. (In particular, $2A - 5H$ is a positive integer.) Inverting gives $\frac{2000}{3} \le \frac{5A}{2A-5H} < 2000$. Multiplying by $\frac{2A-5H}{5}$ gives

$$\frac{400}{3}(2A-5H) \le A < 400(2A-5H).$$

Suppose that $2A - 5H \ge 2$. Then

$$A \ge \frac{400}{3}(2A-5H) \ge \frac{800}{3} = 266\frac{2}{3}.$$

Suppose that $2A - 5H = 1$. Then

$$A \ge \frac{400}{3}(2A-5H) \ge \frac{400}{3} = 133\frac{1}{3}.$$

Also $2A - 5H = 1$ implies that A is 3 mod 5. So $A \ge 138$. We can achieve equality by choosing $A = 138$ and $2A - 5H = 1$ (that is, $H = 55$).

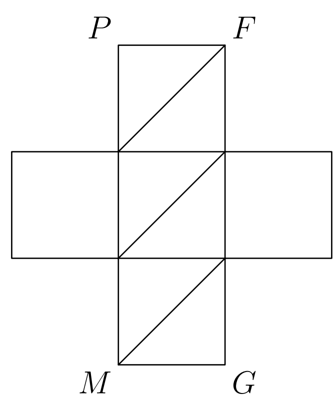
Combining the two paragraphs above, we see that the smallest possible value of A is $\boxed{138}$.

Problem 7 Let n be a positive integer. In n-dimensional space, consider the 2^n points whose coordinates are all ± 1. Imagine placing an n-dimensional ball of radius 1 centered at each of these 2^n points. Let B_n be the largest n-dimensional ball centered at the origin that does not intersect the interior of any of the original 2^n balls. What is the smallest value of n such that B_n contains a point with a coordinate greater than 2?

Answer: 10

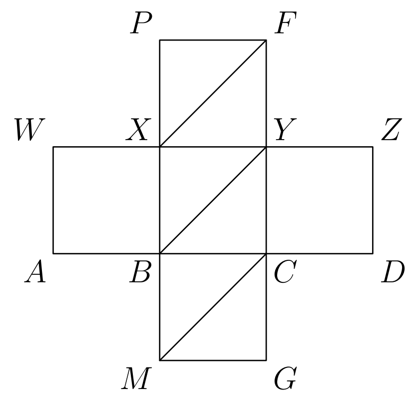
Solution: In n-dimensional space, every point whose coordinates are all $+1$ or -1 is distance $\sqrt{n}$ away from the origin. So a ball of radius 1 centered at such a point will be distance $\sqrt{n} - 1$ away from the origin. Hence the ball B_n has radius $\sqrt{n} - 1$. In particular, the largest coordinate of a point in B_n is $\sqrt{n} - 1$. Thus B_n will contain a point with a coordinate greater than 2 if and only if $\sqrt{n} - 1 > 2$. Solving for n gives $n > 9$. So the smallest possible value of n is $\boxed{10}$.

Problem 8 In the diagram below, how many different routes are there from point M to point P using only the line segments shown? A route is not allowed to intersect itself, not even at a single point.



Answer: 70

Solution: Let's label the vertices of the diagram as follows.



Say that vertices M and G are on the first floor; A, B, C, and D are on the second floor; W, X, Y, and Z are on the third floor; and finally P and F are on the fourth floor.

First, let's count the routes that never go downward. There are 3 segments from the first to the second floor, 5 segments from the second to the third floor, and 3 segments from the third to the fourth floor. Choosing one segment from each of these three categories determines the route from M to P. So there are $3 \cdot 5 \cdot 3 = 45$ such routes.

Second, let's count the routes that go downward from the third to the second floor. The downward step must go from Y to B. There are 2 ways to go from M to C, 2 ways to go from C to Y, 1 way to go from Y to B, 2 ways to go from B to X, and 2 ways to go from X to P. So there are $2 \cdot 2 \cdot 1 \cdot 2 \cdot 2 = 16$ such routes.

Finally, let's count the routes that go downward from the fourth to the third floor. The downward step must go from F to X. There are 3 ways to go from M to the second floor, 3 ways to go from the second floor to Y, 1 way to go from Y to F, 1 way to go from F to X, and 1 way to go from X to P. So there are $3 \cdot 3 \cdot 1 \cdot 1 \cdot 1 = 9$ such routes.

Altogether, these three cases give $45 + 16 + 9 = \boxed{70}$ routes.

Problem 9 Say that a rational number is *special* if its decimal expansion is of the form $0.\overline{abcdef}$, where a, b, c, d, e, and f are digits (possibly equal) that include each of the digits 2, 0, 1, and 5 at least once (in some order). How many special rational numbers are there?

Answer: 23,160

Solution: We need to count the number of choices of $abcdef$ that contain at least one each of 2, 0, 1, and 5. We will do casework based on the number of distinct digits in $abcdef$.

First, suppose that $abcdef$ has six distinct digits. There are $\binom{6}{2} = 15$ ways to choose the two digits aside from 2, 0, 1, and 5. There are $6! = 720$ ways to arrange the six digits. So this case has $15 \cdot 720 = 10{,}800$ choices.

Second, suppose that $abcdef$ has five distinct digits. There are 6 ways to choose the extra digit besides 2, 0, 1, and 5. There are 5 ways to choose which of the five digits appears twice. There are $6!/2! = 360$ ways to arrange the six digits. So this case has $6 \cdot 5 \cdot 360 = 10{,}800$ choices.

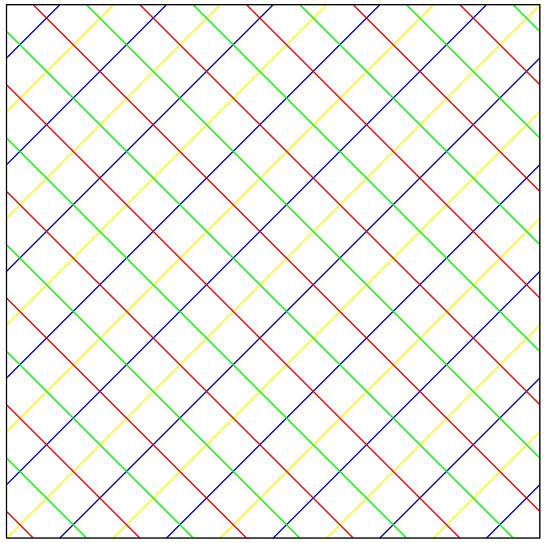
Third, suppose that $abcdef$ has four distinct digits, which must be 2, 0, 1, and 5. Either one of these digits appears three times or two of these digits appear twice. In the first subcase, we have 4 ways to choose the triplicated digit and $6!/3! = 120$ ways to arrange the six digits, giving $4 \cdot 120 = 480$ choices. In the second subcase, we have $\binom{4}{2} = 6$ ways to choose the two duplicated digits and $6!/(2! \cdot 2!) = 180$ ways to arrange the six digits, giving $6 \cdot 180 = 1080$ choices. So the third case has $480 + 1080 = 1560$ choices.

Adding up the three cases gives $10{,}800 + 10{,}800 + 1560 = \boxed{23{,}160}$ choices.

Problem 10 Among all pairs of real numbers (x, y) such that $\sin \sin x = \sin \sin y$ with $-10\pi \le x, y \le 10\pi$, Oleg randomly selected a pair (X, Y). Compute the probability that $X = Y$. Express your answer as a fraction in simplest form.

Answer: $\dfrac{1}{20}$

Solution: The sin function is increasing on $[-\pi/2, \pi/2]$ and hence increasing on $[-1, 1]$. Because $\sin x$ and $\sin y$ are in $[-1, 1]$, the given equation $\sin \sin x = \sin \sin y$ is equivalent to $\sin x = \sin y$. In turn, $\sin x = \sin y$ is equivalent to $y = x + 2k\pi$ or $y = (2k+1)\pi - x$ for some integer k. Because x and y are in $[-10\pi, 10\pi]$, these linear equations are the same as $y \equiv x + 2k\pi \pmod{20\pi}$ or $y \equiv (2k+1)\pi - x \pmod{20\pi}$ for some integer k from 0 through 9. Each of these 20 congruences is a "line segment" of the same length. (Here each "line segment" is allowed to wrap around because of the $\bmod 20\pi$ condition.) Below is a graph of these 20 line segments.



Every pair of these line segments intersect in at most one point. So the total length of the union of these 20 line segments is 20 times the length of one of the segments. The condition $X = Y$ corresponds to the line segment $y \equiv x \pmod{20\pi}$, which is one of the 20 line segments. So the probability that $X = Y$ is $\boxed{1/20}$.

Problem 11 Let $A = (2, 0)$, $B = (0, 2)$, $C = (-2, 0)$, and $D = (0, -2)$. Compute the greatest possible value of the product $PA \cdot PB \cdot PC \cdot PD$, where P is a point on the circle $x^2 + y^2 = 9$.

Answer: 97

Solution: We will interpret each point as a complex number. With that interpretation, $A = 2$, $B = 2i$, $C = -2$, and $D = -2i$. Also $|P| = 3$. The distance PA is $|P - A|$, which is $|P - 2|$. Similarly, PB is $|P - 2i|$, PC is $|P + 2|$, and PD is $|P + 2i|$. So $PA \cdot PC$ is $|P - 2| \cdot |P + 2|$, which is $|P^2 - 4|$. Similarly, $PB \cdot PD$ is $|P - 2i| \cdot |P + 2i|$, which is $|P^2 + 4|$. Hence $PA \cdot PB \cdot PC \cdot PD$ is $|P^2 - 4| \cdot |P^2 + 4|$, which is $|P^4 - 16|$. By the triangle inequality,

$$|P^4 - 16| \leq |P^4| + 16 = |P|^4 + 16 = 3^4 + 16 = 81 + 16 = 97.$$

If $P^4 = -81$, then the inequality becomes an equality. So the greatest possible value of $PA \cdot PB \cdot PC \cdot PD$ is $\boxed{97}$.

Problem 12 A *permutation* of a finite set is a one-to-one function from the set onto itself. A *cycle* in a permutation P is a nonempty sequence of distinct items $x_1, \ldots, x_n$ such that $P(x_1) = x_2$, $P(x_2) = x_3$, $\ldots$, $P(x_n) = x_1$. Note that we allow the 1-cycle x_1 where $P(x_1) = x_1$ and the 2-cycle x_1, x_2 where $P(x_1) = x_2$ and $P(x_2) = x_1$. Every permutation of a finite set splits the set into a finite number of disjoint cycles. If this number equals 2, then the permutation is called *bi-cyclic*. Compute the number of bi-cyclic permutations of the 7-element set formed by the letters of "PROBLEM".

Answer: 1764

Solution: We can split a 7-element set into two nonempty sets in three ways: a 4–3 split, a 5–2 split, and a 6–1 split. Let's count the 4–3 case first. We have $\binom{7}{3}$ ways to split the 7 elements. Then the number of 4-cycles is 3!. (After fixing one element of the 4-cycle, the other three elements can be listed in 3! ways.) Similarly, the number of 3-cycles is 2!. So the 4–3 case has $\binom{7}{3} \cdot 3! \cdot 2!$ permutations, which is $35 \cdot 6 \cdot 2$, or 420.

We can count the other cases similarly. The 5–2 split has $\binom{7}{2} \cdot 4! \cdot 1!$ permutations, which is $21 \cdot 24 \cdot 1$, or 504. The 6–1 split has $\binom{7}{1} \cdot 5! \cdot 0!$ permutations, which is $7 \cdot 120 \cdot 1$, or 840. So the total number of bi-cyclic permutations is $420 + 504 + 840$, which is $\boxed{1764}$.

Problem 13 Joel selected an acute angle x (strictly between 0 and 90 degrees) and wrote the values of $\sin x$, $\cos x$, and $\tan x$ on three different cards. Then he gave those cards to three students, Malvina, Paulina, and Georgina, one card to each, and asked them to figure out which trigonometric function (sin, cos, or tan) produced their cards. Even after sharing the values on their cards with each other, only Malvina was able to surely identify which function produced the value on her card. Compute the sum of all possible values that Joel wrote on Malvina's card. Express your answer in simplified radical form.

Answer: $\dfrac{1 + \sqrt{5}}{2}$

Solution: The functions sin, cos, and tan are each one-to-one on the interval $(0, \pi/2)$. Hence, because Malvina's function is known, the value of angle x is known. In particular, the values of $\sin x$, $\cos x$, and $\tan x$ are each known. Because Paulina's function and Georgina's function are not known, Paulina's value and Georgina's value must be the same.

Let's do case analysis based on Malvina's function. First, assume that Malvina's function is tan. Then $\sin x = \cos x$. Hence $\tan x = 1$. So Malvina's value is 1.

Second, assume that Malvina's function is cos. Then $\tan x = \sin x$. Because x is acute, $\sin x < \tan x$. So we have reached a contradiction. Hence, Malvina's function can't be cos.

Third, assume that Malvina's function is sin. Then $\tan x = \cos x$. Multiplying both sides by $\cos x$ gives $\sin x = \cos^2 x = 1 - \sin^2 x$. So $\sin^2 x + \sin x - 1 = 0$. By the quadratic formula, $\sin x = (-1 \pm \sqrt{5})/2$. Because $\sin x$ is between -1 and 1, the only solution is $\sin x = (-1 + \sqrt{5})/2$. So Malvina's value is $(-1 + \sqrt{5})/2$.

Altogether, Malvina's possible values are 1 and $(-1 + \sqrt{5})/2$. Their sum is $\boxed{(1 + \sqrt{5})/2}$.

Problem 14 Let C be a three-dimensional cube with edge length 1. There are 8 equilateral triangles whose vertices are vertices of C. The 8 planes that contain these 8 equilateral triangles divide C into several nonoverlapping regions. Find the volume of the region that contains the center of C. Express your answer as a fraction in simplest form.

Answer: $\dfrac{1}{6}$

Solution: First, let's imagine the 8 equilateral triangles. Let V be one of the 8 vertices of cube C. Consider the three vertices X, Y, and Z that are adjacent to (share an edge with) V. Let T be the triangle XYZ. Then T is equilateral. So to each vertex V corresponds an equilateral triangle $T(V)$. That gives the 8 equilateral triangles. Each side of $T(V)$ is a diagonal of one of the three faces that contain V. In particular, $T(V)$ contains the center of each of the three faces that contains V.

Next, let's consider the intersection of two of the triangles $T(V)$ and $T(W)$. By the previous paragraph, the intersection contains the center of each face that contains vertices V and W. If V and W are adjacent vertices, then there are two such faces (the faces that contain the edge $\overline{VW}$).

Now we can determine the region that contains the center of cube C. Between every two adjacent faces, draw the segment between their centers. This will form an octahedron whose 6 vertices are the centers of the 6 faces of the cube. By our previous work, that is the region that contains the center of the cube. To compute the volume of the octahedron, divide it into two square pyramids. The base of each pyramid is a square with side length $\sqrt{2}/2$; hence the base has area $1/2$. The height of each pyramid is $1/2$. So the volume of each pyramid is $(1/3)(1/2)(1/2) = 1/12$. Hence the volume of the octahedron is $2(1/12) = \boxed{1/6}$.

Problem 15 Let z_1, z_2, z_3, and z_4 be the four distinct complex solutions of the equation

$$z^4 - 6z^2 + 8z + 1 = -4(z^3 - z + 2)i.$$

Find the sum of the six pairwise distances between z_1, z_2, z_3, and z_4. Express your answer in simplified radical form.

Answer: $6\sqrt{3} + 6$

Solution: Let's bring all the terms of the equation to the left side:

$$z^4 + 4iz^3 - 6z^2 + (8 - 4i)z + (1 + 8i) = 0.$$

The first three terms match those of the fourth power

$$(z + i)^4 = z^4 + 4iz^3 - 6z^2 - 4iz + 1.$$

So the previous equation simplifies to

$$(z + i)^4 + 8z + 8i = 0.$$

Let $w = z + i$. Then the equation simplifies further to $w^4 + 8w = 0$. We can factor out w to get $w(w^3 + 8) = 0$. By the sum of cubes, we have

$$w^3 + 8 = w^3 + 2^3 = (w + 2)(w^2 - 2w + 4).$$

So our equation becomes

$$w(w + 2)(w^2 - 2w + 4) = 0.$$

The roots of this equation are 0, -2, $1 + \sqrt{3}i$, and $1 - \sqrt{3}i$. The distance between 0 and each of the other roots is 2. The distance between each pair of the nonzero roots is $2\sqrt{3}$. So the pairwise distances between the roots of w add up to $6\sqrt{3} + 6$. The roots of z are just translations of the roots of w, so the pairwise distances between the roots of z also add up to $\boxed{6\sqrt{3} + 6}$.

Problem 16 An ant begins at a vertex of a convex regular icosahedron (a figure with 20 triangular faces and 12 vertices). The ant moves along one edge at a time. Each time the ant reaches a vertex, it randomly chooses to next walk along any of the edges extending from that vertex (including the edge it just arrived from). Find the probability that after walking along exactly six (not necessarily distinct) edges, the ant finds itself at its starting vertex. Express your answer as a fraction in simplest form.

Answer: $\dfrac{273}{3125}$

Solution: Let A be the starting vertex. From A, there are 5 vertices that are distance 1 away, 5 vertices that are distance 2 away, and 1 vertex that is distance 3 away. (By "distance", we mean the fewest number of edges on a path.) After one step of the random walk, the ant will be distance 1 away from A with probability 1. After two steps, the ant will be distance 0 away (from A) with probability $1/5$, will be distance 1 away with probability $2/5$, and will be distance 2 away with probability $2/5$. After three steps, the ant will be distance 0 away with probability $(2/5)(1/5) = 2/25$, will be distance 1 away with probability $(1/5)(1) + (2/5)(2/5) + (2/5)(2/5) = 13/25$, will be distance 2 away with probability $(2/5)(2/5) + (2/5)(2/5) = 8/25$, and will be distance 3 away with probability $(2/5)(1/5) = 2/25$.

We could keep going like this, but let's take a shortcut. After three steps, the ant will be at some vertex, call it B. To end at A after six steps, the ant must go from B to A in the final three steps. We can use our work for the first three steps to figure out the probability of this event. The probability of going from B to A in the final three steps is

$$\left(\frac{2}{25}\right)^2 + \frac{1}{5}\left(\frac{13}{25}\right)^2 + \frac{1}{5}\left(\frac{8}{25}\right)^2 + \left(\frac{2}{25}\right)^2 = \frac{4}{625} + \frac{169}{3125} + \frac{64}{3125} + \frac{4}{625} = \frac{273}{3125}\,.$$

So the probability of returning to A after six steps is $\boxed{273/3125}$.

Problem 17 Let S be the sum of all distinct real solutions of the equation

$$\sqrt{x + 2015} = x^2 - 2015.$$

Compute $\lfloor 1/S \rfloor$. Recall that if r is a real number, then $\lfloor r \rfloor$ (the *floor* of r) is the greatest integer that is less than or equal to r.

Answer: 89

Solution: Because x is real, the left-hand side is nonnegative, which means the right-hand side is nonnegative, which means that $|x| \geq \sqrt{2015}$. By squaring both sides, we get the equation $(x^2 - 2015)^2 - 2015 = x$. Let $P(x)$ be the quadratic $x^2 - 2015$. Then the equation becomes $P(P(x)) = x$, or $P(P(x)) - x = 0$. If $P(x) = x$, then $P(P(x)) = P(x) = x$. In other words, the quadratic $P(x) - x$ is a factor of the quartic $P(P(x)) - x$. By dividing, we can find the other quadratic factor of $P(P(x)) - x$. Namely, $P(P(x)) - x = (P(x) - x)(x^2 + x - 2014)$. By the quadratic formula, the two roots of $P(x) - x$ are $(1 \pm \sqrt{8061})/2$. The two roots of $x^2 + x - 2014$ are $(-1 \pm \sqrt{8057})/2$. Of these four roots, the only ones that have absolute value at least $\sqrt{2015}$ are $(1 + \sqrt{8061})/2$ and $(-1 - \sqrt{8057})/2$. So these are the two real roots of the original equation. Their sum S is $(\sqrt{8061} - \sqrt{8057})/2$. So the reciprocal of S is

$$\frac{1}{S} = \frac{2}{\sqrt{8061} - \sqrt{8057}} = \frac{\sqrt{8061} + \sqrt{8057}}{2}.$$

Because $\sqrt{8061}$ and $\sqrt{8057}$ are strictly between 89 and 90, the floor of $1/S$ is $\boxed{89}$.

Problem 18 Let n be a positive integer. When the leftmost digit of (the standard base 10 representation of) n is shifted to the rightmost position (the units position), the result is $n/3$. Find the smallest possible value of the sum of the digits of n.

Answer: 126

Solution: Suppose that n has d digits. Suppose that q is the leftmost digit of n. Then $n = 10^{d-1}q + r$, where r is n without its leftmost digit. So r is $n - 10^{d-1}q$. Shifting q to the rightmost position results in $10r + q$. So we get the equation $n/3 = 10r + q$. Multiplying by 3 gives $n = 30r + 3q$. Using our previous formula for r gives $n = 30(n - 10^{d-1}q) + 3q$. Solving for n gives $n = 3q(10^d - 1)/29$. In particular, 10^d is 1 mod 29.

By Fermat's Little Theorem, 10^{28} is 1 mod 29. Let k be the smallest positive integer such that 10^k is 1 mod 29. It's easy to see that if 10^j is 1 mod 29, then j is a multiple of k. In particular, k is a divisor of 28. To figure out k, let's test all the proper divisors of 28.

$$\begin{aligned}
10^1 &\equiv 10 \pmod{29} \\
10^2 &\equiv 13 \pmod{29} \\
10^4 &\equiv 24 \pmod{29} \\
10^7 &\equiv 17 \pmod{29} \\
10^{14} &\equiv 28 \equiv -1 \pmod{29}
\end{aligned}$$

None of these proper divisors lead to 1 mod 29. So k is 28. Because 10^d is 1 mod 29, that means d is a multiple of 28.

Let's handle the case $d = 28$ first. Recall that $n = 3q(10^{28} - 1)/29$. By difference of squares, $n = 3q(10^{14} - 1)(10^{14} + 1)/29$. Because 10^{14} is -1 mod 29, we know that $3q(10^{14}+1)/29$ is an integer; in fact, it is an integer with exactly 14 digits. Multiplying by $10^{14} - 1$ gives a 28-digit number in which the second half is the nines complement of the first half. So n consists of 14 pairs of digits that add up to 9. Hence the sum of the digits of n is $14 \cdot 9$, which is 126.

Finally, let's handle the more general case when d is a multiple of 28. Recall that $n = 3q(10^d - 1)/29$. By difference of powers, we can write n as

$$\frac{3q(10^d - 1)}{29} \cdot \left(10^{d-28} + 10^{d-56} + \cdots + 10^{28} + 10^0\right).$$

In the previous paragraph, we showed that the first factor is a 28-digit number with a digit sum of 126. Multiplying by

$$10^{d-28} + 10^{d-56} + \cdots + 10^{28} + 10^0$$

repeats the same 28-digit number $d/28$ times. So the digit sum of n is $126(d/28)$. In particular, the digit sum is at least 126. So the smallest possible digit sum of n is $\boxed{126}$.

Problem 19 Sabrina has a fair tetrahedral die whose faces are numbered 1, 2, 3, and 4, respectively. She creates a sequence by rolling the die and recording the number on its bottom face. However, she discards (without recording) any roll such that appending its number to the sequence would result in two consecutive terms that sum to 5. Sabrina stops the moment that all four numbers appear in the sequence. Find the expected (average) number of terms in Sabrina's sequence.

Answer: 10

Solution: Suppose we have a biased coin with probability p of coming up heads. Suppose we keep tossing the coin until it comes up heads. What is the expected number of coin tosses? The answer is $1/p$. This is a well-known result that we will use twice below.

We will split Sabrina's sequence into four stages. The first stage is the part up to (and including) the first distinct term. After that, the second stage goes up to the second distinct term. After that, the third stage goes up to the third distinct term. After that, the fourth stage goes up to the fourth distinct term.

The first stage has exactly 1 term. In the second stage, every term has three possible values, two of which will end the stage. So the second stage is like the biased coin problem with $p = 2/3$. Hence the expected number of terms in the second stage is $3/2$.

In the third stage, every term has three possible values, only one of which will end the stage. So the third stage is like the biased coin problem with $p = 1/3$. Hence the expected number of terms in the third stage is 3.

The fourth stage is trickier. Let a be the expected number of terms in the fourth stage. Without loss of generality, the first term in Sabrina's sequence is 1 and the second distinct term is 2. For concreteness, suppose the third distinct term is 3. Hence the first term in the fourth stage will be 1, 3, or 4. A 3 will just repeat the situation and a 4 will end the stage. If the first term in the fourth stage is 1, let b be the expected number of terms after that. By considering the three possible first terms of the fourth stage, we get the equation

$$a = \frac{1}{3} \cdot b + \frac{1}{3} \cdot a + \frac{1}{3} \cdot 0 + 1.$$

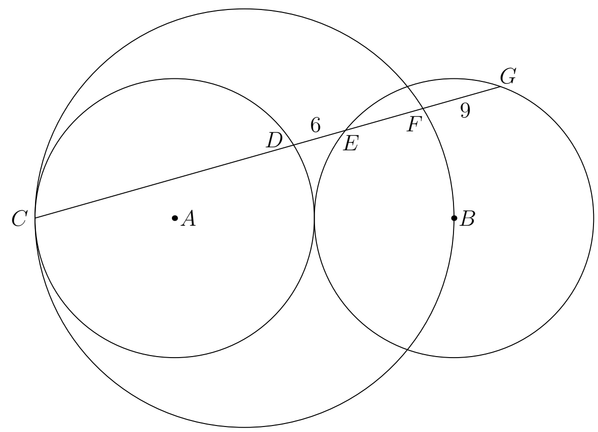
Simplifying gives $2a = b + 3$. If the first term in the fourth stage is 1, then the next term will be 1, 2, or 3. A 1 repeats the current situation and a 2 or 3 goes back to the original situation. So we get the equation

$$b = \frac{1}{3} \cdot b + \frac{2}{3} \cdot a + 1.$$

Simplifying gives $b = a + 3/2$. Solving our two equations gives $a = 9/2$ and $b = 6$. So the expected number of terms in the fourth stage is $9/2$.

Adding all four stages, we see that the expected number of terms in Sabrina's sequence is $1 + 3/2 + 3 + 9/2$, which is $\boxed{10}$.

Problem 20 In the diagram below, the circle with center A is congruent to and tangent to the circle with center B. A third circle is tangent to the circle with center A at point C and passes through point B. Points C, A, and B are collinear. The line segment $\overline{CDEFG}$ intersects the circles at the indicated points. Suppose that $DE = 6$ and $FG = 9$. Find AG. Express your answer in simplified radical form.



Answer: $9\sqrt{19}$

Solution: Because angle BFC is inscribed in a semicircle, it is a right angle. Because $BE = BG$, it follows that $EF = FG = 9$. Let P be the point where the two small circles intersect. Because angle PDC is also inscribed in a semicircle, it is a right angle. Because $\angle PDC = \angle BFC$, lines $\overline{PD}$ and $\overline{BF}$ are parallel. By triangle proportionality, $CD/DF = CP/PB = 2$. Hence

$$CD = 2 \cdot DF = 2 \cdot 15 = 30.$$

Let Q be the point opposite point P on the circle with center B. Let's apply Power of a Point to point C and the circle with center B. We get the equation

$$CP \cdot CQ = CE \cdot CG = 36 \cdot 54.$$

Let r be the radius of the two small circles. Because $CP = 2r$ and $CQ = 4r$, the equation becomes

$$2r \cdot 4r = 36 \cdot 54.$$

Hence $r^2 = 243$.

Finally, let's apply Power of a Point to point G and the circle with center A. We get the equation

$$CG \cdot DG = (AG + r)(AG - r) = AG^2 - r^2.$$

Solving for AG^2, we get

$$AG^2 = CG \cdot DG + r^2 = 54 \cdot 24 + 243 = 81(16 + 3) = 81 \cdot 19.$$

Hence $AG = \sqrt{81 \cdot 19} = \boxed{9\sqrt{19}}$.

Credits

The authors of the 20 problems are as follows.

Ravi Boppana: 4, 6, 7
Mathew Crawford: 9, 14, 15, 16, 18, 19, 20
Oleg Kryzhanovsky: 5, 8, 10, 11, 12, 13, 17
Joseph Woo: 1, 2, 3