The Advantage Testing Foundation
Math Prize for Girls Olympiad

Wednesday, November 16, 2016

Time Limit: 4 hours

1. Triangle T_1 has sides of length a_1, b_1, and c_1; its area is K_1. Triangle T_2 has sides of length a_2, b_2, and c_2; its area is K_2. Triangle T_3 has sides of length $a_1 + a_2$, $b_1 + b_2$, and $c_1 + c_2$; its area is K_3.

(a) Prove that $K_1^2 + K_2^2 < K_3^2$.

(b) Prove that $\sqrt{K_1} + \sqrt{K_2} \le \sqrt{K_3}$.

2. Eve picked some apples, each weighing at most $\frac{1}{2}$ pound. Her apples weigh a total of W pounds, where $W > \frac{1}{3}$. Prove that she can place all her apples into $\left\lceil \frac{3W-1}{2} \right\rceil$ or fewer baskets, each of which holds up to 1 pound of apples. (The apples are not allowed to be cut into pieces.) Note: If x is a real number, then $\lceil x \rceil$ (the ceiling of x) is the least integer that is greater than or equal to x.

3. Let n be a positive integer. Let $x_1, x_2, \ldots, x_n$ be a sequence of n real numbers. Say that a sequence $a_1, a_2, \ldots, a_n$ is *unimodular* if each a_i is ± 1. Prove that

$$\sum a_1 a_2 \ldots a_n (a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)^n = 2^n n!\, x_1 x_2 \ldots x_n,$$

where the sum is over all 2^n unimodular sequences $a_1, a_2, \ldots, a_n$.

4. Let $d(n)$ be the number of positive divisors of a positive integer n. Let $\mathbb{N}$ be the set of all positive integers. Say that a bijection F from $\mathbb{N}$ to $\mathbb{N}$ is *divisor-friendly* if $d(F(mn)) = d(F(m))d(F(n))$ for all positive integers m and n. (Note: A bijection is a one-to-one, onto function.) Does there exist a divisor-friendly bijection? Prove or disprove.