The Advantage Testing Foundation

Math Prize for Girls at MIT

2016 Olympiad Solutions

Problem 1 Triangle T_1 has sides of length a_1, b_1, and c_1; its area is K_1. Triangle T_2 has sides of length a_2, b_2, and c_2; its area is K_2. Triangle T_3 has sides of length $a_1 + a_2$, $b_1 + b_2$, and $c_1 + c_2$; its area is K_3.

(a) Prove that $K_1^2 + K_2^2 < K_3^2$.

(b) Prove that $\sqrt{K_1} + \sqrt{K_2} \le \sqrt{K_3}$.

Solution: Let $a_3 = a_1 + a_2$, let $b_3 = b_1 + b_2$, and let $c_3 = c_1 + c_2$. Then triangle T_i has sides of length a_i, b_i, and c_i (for i from 1 to 3). Let s_i be the semiperimeter $(a_i + b_i + c_i)/2$; note that $s_3 = s_1 + s_2$. Let $p_i = s_i - a_i$, let $q_i = s_i - b_i$, and let $r_i = s_i - c_i$; note that $p_3 = p_1 + p_2$, $q_3 = q_1 + q_2$, and $r_3 = r_1 + r_2$. All these numbers are positive by the triangle inequality. Heron's formula says that the area K_i is $\sqrt{p_i q_i r_i s_i}$.

(a) From our work above, we have

$$\begin{aligned} K_3^2 &= p_3 q_3 r_3 s_3 \\ &= (p_1 + p_2)(q_1 + q_2)(r_1 + r_2)(s_1 + s_2) \\ &> p_1 q_1 r_1 s_1 + p_2 q_2 r_2 s_2 \\ &= K_1^2 + K_2^2. \end{aligned}$$

(b) By applying the Cauchy-Schwarz inequality three times, we have

$$\begin{aligned} \sqrt{K_1} + \sqrt{K_2} &= (p_1 q_1 r_1 s_1)^{1/4} + (p_2 q_2 r_2 s_2)^{1/4} \\ &= (p_1 q_1)^{1/4}(r_1 s_1)^{1/4} + (p_2 q_2)^{1/4}(r_2 s_2)^{1/4} \\ &\le \sqrt{(p_1 q_1)^{1/2} + (p_2 q_2)^{1/2}}\sqrt{(r_1 s_1)^{1/2} + (r_2 s_2)^{1/2}} \\ &\le \left[(p_1 + p_2)^{1/4}(q_1 + q_2)^{1/4}\right]\left[(r_1 + r_2)^{1/4}(s_1 + s_2)^{1/4}\right] \\ &= (p_3 q_3 r_3 s_3)^{1/4} \\ &= \sqrt{K_3}\,. \end{aligned}$$

> **Alternative Solution:** Instead of the Cauchy-Schwarz inequality, we could have used the AM-GM inequality or the generalized Hölder's inequality.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 2 Eve picked some apples, each weighing at most $\frac{1}{2}$ pound. Her apples weigh a total of W pounds, where $W > \frac{1}{3}$. Prove that she can place all her apples into $\lceil \frac{3W-1}{2} \rceil$ or fewer baskets, each of which holds up to 1 pound of apples. (The apples are not allowed to be cut into pieces.) Note: If x is a real number, then $\lceil x \rceil$ (the ceiling of x) is the least integer that is greater than or equal to x.

Solution: We will start with the following lemma about packing one basket.

Lemma 1 *If $W > \frac{2}{3}$, then there is a subset of the apples that weighs more than $\frac{2}{3}$ pounds and at most 1 pound.* □

PROOF Because $W > \frac{2}{3}$, there is more than one apple. Say the two heaviest apples weigh x and y pounds, respectively. If $x + y > \frac{2}{3}$, then we're done: choose the two heaviest apples. Otherwise, $x + y \leq \frac{2}{3}$. Every other apple, being no heavier than the first two, weighs at most $\frac{1}{3}$ pound. Add the two heaviest apples to the subset, and then keep adding apples to the subset until it first exceeds $\frac{2}{3}$ pounds. The final apple weighs at most $\frac{1}{3}$ pound and the previous apples of the subset weigh at most $\frac{2}{3}$ pounds. Hence the subset weighs at most 1 pound. ■

We claim that for every positive integer b, if $W \leq \frac{2b+1}{3}$, then Eve can pack all her apples in at most b baskets (each of which holds up to 1 pound of apples). This claim solves the problem, by choosing b to be the ceiling of $\frac{3W-1}{2}$. We will prove the claim by induction on b. The base case, $b = 1$, is trivial: if $W \leq 1$, then Eve can pack all her apples in 1 basket. By induction, suppose the claim is true for $b - 1$; we will prove it for b.

If $W \leq \frac{2}{3}$, then we're done: pack all the apples in one basket. Otherwise, $W > \frac{2}{3}$. By the Lemma, there is a subset of the apples that weighs V pounds, where $\frac{2}{3} < V \leq 1$. Pack this subset of apples in one basket. Note that

$$W - V < W - \frac{2}{3} \leq \frac{2b+1}{3} - \frac{2}{3} = \frac{2b-1}{3}.$$

So by induction, we can pack all the apples not in the subset in $b-1$ baskets. Hence we can pack all the apples in b baskets. That completes the inductive proof of the claim, and we're done.

Problem 3 Let n be a positive integer. Let x_1, x_2, …, x_n be a sequence of n real numbers. Say that a sequence a_1, a_2, …, a_n is *unimodular* if each a_i is ± 1. Prove that

$$\sum a_1 a_2 \dots a_n (a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)^n = 2^n n!\, x_1 x_2 \dots x_n,$$

where the sum is over all 2^n unimodular sequences a_1, a_2, …, a_n.

Solution: An expression of the form $(y_1 + y_2 + \cdots + y_j)^n$ can be expanded into the sum

$$\sum_f y_{f(1)} y_{f(2)} \cdots y_{f(n)} ,$$

where the sum is over all functions f from $\{1, \dots, n\}$ to $\{1, \dots, j\}$. In particular, we have

$$(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)^n = \sum_f a_{f(1)} \dots a_{f(n)} x_{f(1)} \dots x_{f(n)} ,$$

where the sum is over all functions f from $\{1, \dots, n\}$ to itself. So the sum we're interested in is

$$\begin{aligned} &\sum_a a_1 \dots a_n (a_1 x_1 + \cdots + a_n x_n)^n \\ &\qquad = \sum_a a_1 \dots a_n \sum_f a_{f(1)} \dots a_{f(n)} x_{f(1)} \dots x_{f(n)} \\ &\qquad = \sum_f x_{f(1)} \dots x_{f(n)} \sum_a a_1 \dots a_n a_{f(1)} \dots a_{f(n)}. \end{aligned}$$

We will partition this sum into two cases.

Case 1: f is a permutation. Then we have

$$\begin{aligned} x_{f(1)} \dots x_{f(n)} \sum_a a_1 \dots a_n a_{f(1)} \dots a_{f(n)} &= x_1 \dots x_n \sum_a a_1^2 \dots a_n^2 \\ &= x_1 \dots x_n \sum_a 1 \\ &= 2^n x_1 \dots x_n. \end{aligned}$$

Because there are $n!$ permutations, the permutations contribute $2^n n! x_1 \dots x_n$ to the sum.

Case 2: f is not a permutation. Then some number k in $\{1, \ldots, n\}$ is missing from the image of f. Given a unimodular sequence a, define its k-flip to be the unimodular sequence b that is the same as a except that $b_k = -a_k$. Note that

$$b_1 \ldots b_n b_{f(1)} \ldots b_{f(n)} = -a_1 \ldots a_n a_{f(1)} \ldots a_{f(n)} .$$

In other words, the terms $a_1 \ldots a_n a_{f(1)} \ldots a_{f(n)}$ and $b_1 \ldots b_n b_{f(1)} \ldots b_{f(n)}$ cancel out. The 2^n unimodular sequences can be partitioned into 2^{n-1} pairs of k-flips. So the entire sum

$$\sum_a a_1 \ldots a_n a_{f(1)} \ldots a_{f(n)}$$

vanishes. As a result, the non-permutations contribute nothing to the sum.

Adding the two cases, we see that the total sum is $2^n n!\, x_1 \ldots x_n$.

Alternative Solution: In the first step, we could have used the multinomial theorem.

Problem 4 Let $d(n)$ be the number of positive divisors of a positive integer n. Let $\mathbb{N}$ be the set of all positive integers. Say that a bijection F from $\mathbb{N}$ to $\mathbb{N}$ is *divisor-friendly* if $d(F(mn)) = d(F(m))d(F(n))$ for all positive integers m and n. (Note: A bijection is a one-to-one, onto function.) Does there exist a divisor-friendly bijection? Prove or disprove.

Solution: Yes, there is a divisor-friendly bijection. We will construct one below.

Note that the divisor function d maps 1 to itself and maps $\mathbb{N} - \{1\}$ to $\mathbb{N} - \{1\}$.

Lemma 2 *If k is an integer greater than 1, then its preimage $d^{-1}(k)$ is infinite.* □

PROOF If p is a prime number, then $d(p^{k-1}) = k$. So $d^{-1}(k)$ contains p^{k-1} for every prime p. ■

We will construct our divisor-friendly bijection F inductively. If $n = 1$, then let $F(n)$ be 1. If n is prime, then let $F(n)$ be the smallest integer in $\mathbb{N} - \{F(1), F(2), \ldots, F(n-1)\}$. If n is composite, then the divisor-friendly condition forces $d(F(n))$ to be a particular integer $k > 1$. (Namely, if n is the product of primes $p_1 p_2 \cdots p_i$, then k is the product $d(F(p_1))d(F(p_2)) \cdots d(F(p_i))$.)

Then define $F(n)$ to be some integer in $d^{-1}(k) - \{F(1), F(2), \ldots, F(n-1)\}$. (By the Lemma, the set of choices is infinite and hence nonempty.)

We claim that F is a divisor-friendly bijection. We need to show that F is one-to-one, onto, and divisor-friendly.

By construction, we made $F(n)$ different from all previous values, so F is one-to-one.

Let's show that F is onto. Given a positive integer y, we will show that it is in the image of F. Because F is one-to-one, there is a prime p such that $F(p) > y$. Hence, by our prime construction, y is in $\{F(1), F(2), \ldots, F(p-1)\}$. In particular, y is in the image of F.

Finally, let's show that F is divisor-friendly. If m is the product of primes $p_1 p_2 \cdots p_i$, then $d(F(m))$ is the product $d(F(p_1))d(F(p_2)) \cdots d(F(p_i))$. (If m is 1 or prime, then this identity is trivial. If m is composite, then it's true by our composite construction.) Similarly, if n is the product of primes $q_1 q_2 \cdots q_j$, then $d(F(n))$ is the product $d(F(q_1))d(F(q_2)) \cdots d(F(q_j))$. Similarly, because mn is the product of primes $p_1 \cdots p_i q_1 \cdots q_j$, we know that $d(F(mn))$ is the product $d(F(p_1)) \cdots d(F(p_i))d(F(q_1)) \cdots d(F(q_j))$. So F is divisor-friendly.

Note: This problem was proposed by Oleg Kryzhanovsky.