

THE ADVANTAGE TESTING FOUNDATION



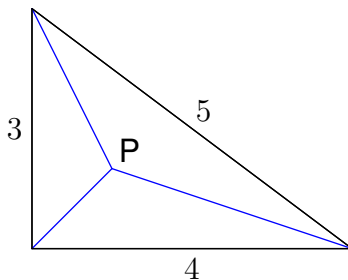
MATH PRIZE *for* GIRLS *at* MIT

2016 SOLUTIONS

Problem 1 Let T be a triangle with side lengths 3, 4, and 5. If P is a point in or on T , what is the greatest possible sum of the distances from P to each of the three sides of T ?

Answer: 4

Solution: Draw segments from point P to each of the vertices of the triangle T , as shown in the figure below.



Let x , y , and z , respectively, be the distances from P to the sides of length 3, 4, and 5. The areas of the three subtriangles are $3x/2$, $4y/2$, and $5z/2$. The area of the original triangle is 6. So we get the equation

$$3x + 4y + 5z = 12.$$

Hence we have

$$x + y + z \leq \frac{3x + 4y + 5z}{3} = \frac{12}{3} = 4.$$

When P is the vertex opposite the side of length 3, we have $x = 4$, $y = 0$, and $z = 0$, which achieves $x + y + z = 4$. So the greatest possible value of $x + y + z$ is $\boxed{4}$.

Problem 2 Katrine has a bag containing 4 buttons with distinct letters M, P, F, G on them (one letter per button). She picks buttons randomly, one at a time, without replacement, until she picks the button with letter G. What is the probability that she has at least three picks and her third pick is the button with letter M? Express your answer as a fraction in simplest form.

Answer: $\frac{1}{12}$

Solution: Even though Katrine stops with G, let's imagine that she picks all 4 buttons no matter what. We'll call this scenario the extended game. The extended game is just a random permutation of the 4 buttons. For the original game to have at least three picks with the third pick being M is equivalent to the extended game having the third pick be M and the fourth pick be G. So in the extended game, the only successful permutations are PFMG and FPMG. Hence the probability of success is $\frac{2}{4!} = \frac{2}{24}$, which simplifies to $\boxed{\frac{1}{12}}$.

Problem 3 Compute the least possible value of $ABCD - AB \times CD$, where $ABCD$ is a 4-digit positive integer, and AB and CD are 2-digit positive integers. (Here A , B , C , and D are digits, possibly equal. Neither A nor C can be zero.)

Answer: 109

Solution: Let x be AB and y be CD . (Note that x and y are between 10 and 99.) Then $ABCD$ is $100x + y$. So $ABCD - AB \times CD$ is $100x + y - xy$. By "completing the product" to factor, we see that

$$100x + y - xy = 100 + (x - 1)(100 - y) \geq 100 + 9 \cdot 1 = 109.$$

Equality is achieved when $x = 10$ and $y = 99$. So the least possible value is $\boxed{109}$.

Problem 4 Compute the smallest positive integer n such that 2016^n does not divide $2016!$.

Answer: 335

Solution: The prime factorization of 2016 is $2^5 \cdot 3^2 \cdot 7$. So 2016^n is $2^{5n} \cdot 3^{2n} \cdot 7^n$. The number of factors of 2 in $2016!$ is

$$1008 + 504 + 252 + 126 + 63 + 31 + 15 + 7 + 3 + 1 = 2010.$$

The number of factors of 3 in $2016!$ is

$$672 + 224 + 74 + 24 + 8 + 2 = 1004.$$

The number of factors of 7 in $2016!$ is

$$288 + 41 + 5 = 334.$$

Because 2016^n does not divide $2016!$, we have $5n > 2010$ or $2n > 1004$ or $n > 334$. Those three inequalities simplify to $n > 402$ or $n > 502$ or $n > 334$. Those three inequalities combine to $n > 334$. So the smallest n is $\boxed{335}$.

Problem 5 A permutation of a finite set S is a one-to-one function from S to S . A permutation P of the set $\{1, 2, 3, 4, 5\}$ is called a W-permutation if $P(1) > P(2) < P(3) > P(4) < P(5)$. A permutation of the set $\{1, 2, 3, 4, 5\}$ is selected at random. Compute the probability that it is a W-permutation. Express your answer as a fraction in simplest form.

Answer: $\frac{2}{15}$

Solution: Say that $P(2)$ and $P(4)$ are the low values; say that $P(1)$, $P(3)$, and $P(5)$ are the high values. It's easy to see that 1 must be low. It's easy to see that 4 and 5 must be high. So the low values are either $\{1, 2\}$ or $\{1, 3\}$.

Let's look at the first case: the low values are $\{1, 2\}$. There are $2! = 2$ ways to assign them to $P(2)$ and $P(4)$. There are $3! = 6$ ways to assign $\{3, 4, 5\}$ to $P(1)$, $P(3)$, and $P(5)$. So there are $2 \cdot 6 = 12$ W-permutations in this case.

Let's now look at the second case: the low values are $\{1, 3\}$. There are $2! = 2$ ways to assign them to $P(2)$ and $P(4)$. For concreteness, suppose that $P(2) = 1$. Then $P(1)$ must be 2. There are $2! = 2$ ways to assign $\{4, 5\}$ to $P(3)$ and $P(5)$. So there are $2 \cdot 2 = 4$ W-permutations in this case.

Altogether, there are $12 + 4 = 16$ W-permutations. So the probability of being a W-permutation is $\frac{16}{5!} = \frac{16}{120}$, which simplifies to $\boxed{\frac{2}{15}}$.

Problem 6 The largest term in the binomial expansion of $(1 + \frac{1}{2})^{31}$ is of the form $\frac{a}{b}$, where a and b are relatively prime positive integers. What is the value of b ? As an example of a binomial expansion, the binomial expansion of an expression of the form $(x + y)^3$ is the sum of four terms

$$x^3 + 3x^2y + 3xy^2 + y^3.$$

Answer: 1024

Solution: The binomial expansion of $(1 + \frac{1}{2})^{31}$ is

$$\sum_{k=0}^{31} \binom{31}{k} \frac{1}{2^k}.$$

Let a_k be the k th term $\binom{31}{k} \frac{1}{2^k}$. We can express a_{k+1} in terms of a_k :

$$a_{k+1} = \binom{31}{k+1} \frac{1}{2^{k+1}} = \frac{31-k}{k+1} \binom{31}{k} \frac{1}{2^{k+1}} = \frac{31-k}{2(k+1)} a_k.$$

So the ratio $\frac{a_{k+1}}{a_k}$ is $\frac{31-k}{2(k+1)}$. This ratio is greater than 1 if $k \leq 9$ and is less than 1 if $k \geq 10$. In particular, a_{10} is the largest term. The value of a_{10} is $\binom{31}{10} \frac{1}{2^{10}}$. Because 31 is 1 less than a power of 2, every binomial coefficient $\binom{31}{k}$ is odd. So the denominator of a_{10} is 2^{10} , which is $\boxed{1024}$.

Problem 7 Let S be the set of all real numbers x such that $0 \leq x \leq 2016\pi$ and $\sin x < 3 \sin(x/3)$. The set S is the union of a finite number of disjoint intervals. Compute the total length of all these intervals. Express your answer in terms of π .

Answer: 1008π

Solution: The triple-angle formula says that

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta.$$

In particular, we have

$$3 \sin(x/3) - \sin x = 4 \sin^3(x/3).$$

So $\sin x < 3 \sin(x/3)$ is equivalent to $\sin(x/3) > 0$. The inequality $\sin(x/3) > 0$ is satisfied on the open intervals $(0, 3\pi)$, $(6\pi, 9\pi)$, \dots , $(2010\pi, 2013\pi)$. There are $2016/6 = 336$ such intervals, each of length 3π . So their total length is $336 \cdot 3\pi$, which is $\boxed{1008\pi}$.

Problem 8 A *strip* is the region between two parallel lines. Let A and B be two strips in a plane. The intersection of strips A and B is a parallelogram P . Let A' be a rotation of A in the plane by 60° . The intersection of strips A' and B is a parallelogram with the same area as P . Let x° be the measure (in degrees) of one interior angle of P . What is the greatest possible value of the number x ?

Answer: 150

Solution: Let h_A be the width of strip A (the distance between its two parallel lines). Let h_B be the width of strip B . The parallelogram P has sides of lengths $h_A/\sin x^\circ$ and $h_B/\sin x^\circ$. In particular, the area of P is $h_A h_B/\sin x^\circ$. When A is rotated, the angle of measure x° becomes an angle of measure say y° . So the area of the new parallelogram is $h_A h_B/\sin y^\circ$. Because the area stayed the same, we have $\sin x^\circ = \sin y^\circ$. Hence y is either x or $180 - x$.

When the strip A is rotated by 60° , what happens to the angle of measure x° ? If $0 < x \leq 60$, then the new angle has measure $(60 - x)^\circ$ or $(x + 60)^\circ$, depending on the direction of the rotation; for the new angle to be x° or $(180 - x)^\circ$, either $x = 30$ or $x = 60$. If $60 \leq x \leq 120$, then the new angle has measure $(x - 60)^\circ$ or $(x + 60)^\circ$; for the new angle to be x° or $(180 - x)^\circ$, either $x = 60$ or $x = 120$. If $120 \leq x < 180$, then the new angle has measure $(x - 60)^\circ$ or $(x - 120)^\circ$; for the new angle to be x° or $(180 - x)^\circ$, either $x = 120$ or $x = 150$.

Among these four possible values of x , the greatest is $\boxed{150}$.

Problem 9 How many distinct lines pass through the point $(0, 2016)$ and intersect the parabola $y = x^2$ at two lattice points? (A lattice point is a point whose coordinates are integers.)

Answer: 36

Solution: The vertical line through $(0, 2016)$ intersects the parabola $y = x^2$ at only one point. A non-vertical line through $(0, 2016)$ intersects the parabola at two points, one on the left and one on the right. Call the two intersection points $(-a, a^2)$ and (b, b^2) , where a and b are positive. Note that the equation

$$y = x^2 - (x + a)(x - b)$$

is a linear equation that passes through $(-a, a^2)$ and (b, b^2) . So it must be the equation of the given line. Plugging in $x = 0$ and $y = 2016$ gives $ab = 2016$.

To get lattice points, we are looking for positive integers a and b such that $ab = 2016$. The prime factorization of 2016 is $2^5 \cdot 3^2 \cdot 7^1$. So 2016 has $6 \cdot 3 \cdot 2 = 36$ positive divisors. Each divisor is a choice for a , which determines b . So the number of solutions is $\boxed{36}$.

Problem 10 How many solutions of the equation $\tan x = \tan \tan x$ are on

the interval $0 \leq x \leq \tan^{-1} 942$? (Here \tan^{-1} means the inverse tangent function, sometimes written \arctan .)

Answer: 300

Solution: Two angles have the same tangent if and only if their difference is a multiple of π . So $\tan x = \tan y$ if and only if $\tan x - y$ is a multiple of π . Let T be the function defined by $T(x) = \tan x - x$. We are looking for solutions to $T(x) = n\pi$, where n is an integer. We will show later that T is strictly increasing on the interval $[0, \pi/2)$. On that interval, T goes from 0 to infinity. So for every nonnegative integer n , there is a unique x in that interval such that $T(x) = n\pi$.

We know that $3.14 < \pi < 3.145$. Multiplying by 300, we get $942 < 300\pi < 943.5$. So

$$T(\tan^{-1} 942) = 942 - \tan^{-1} 942 < 942 < 300\pi.$$

Also

$$T(\tan^{-1} 942) = 942 - \tan^{-1} 942 > 942 - \frac{\pi}{2} > (300\pi - 1.5) - \frac{\pi}{2} > 299\pi.$$

Hence $T(\tan^{-1} 942)$ is strictly between 299π and 300π . So for every integer n , the equation $T(x) = n\pi$ has a solution on the interval $[0, \tan^{-1} 942]$ if and only if $0 \leq n < 300$. There are 300 such integers n , so the number of solutions to our original equation is 300.

We promised to show that T is strictly increasing on the interval $[0, \pi/2)$. We will use the well-known inequality (easy to prove with the unit circle) that $\tan \theta > \theta$ if $0 < \theta < \pi/2$. Let x and y be real numbers such that $0 \leq x < y < \pi/2$. By the tangent subtraction formula, we have

$$y - x < \tan(y - x) = \frac{\tan y - \tan x}{1 + \tan x \tan y} \leq \tan y - \tan x.$$

By rearranging, we see that $T(x) < T(y)$. So T indeed is strictly increasing.

Problem 11 Compute the number of ordered pairs of complex numbers (u, v) such that $uv = 10$ and such that the real and imaginary parts of u and v are integers.

Answer: 48

Solution: Because u and v have integer parts, $|u|^2$ and $|v|^2$ are nonnegative integers. From $uv = 10$, it follows that $|u|^2 \cdot |v|^2 = 100$. So $|u|^2$ and $|v|^2$ are positive integers whose product is 100. We will divide the count into three cases: $|u| < |v|$, $|u| = |v|$, and $|u| > |v|$.

Let's handle the case $|u| < |v|$ first. In that case, $|u|^2$ is a small divisor of 100: either 1, 2, 4, or 5. If $|u|^2 = 1$, then we have 4 choices for u : either ± 1 or $\pm i$. If $|u|^2 = 2$, then we have 4 choices: $\pm 1 \pm i$. If $|u|^2 = 4$, then we have 4 choices: ± 2 or $\pm 2i$. If $|u|^2 = 5$, then we have 8 choices: $\pm 1 \pm 2i$ or $\pm 2 \pm i$. Altogether, we have 20 choices for u . Each such choice gives a single valid choice for v , namely $v = 10/u = 10\bar{u}/|u|^2$. So we have 20 pairs in the case $|u| < |v|$.

Let's next handle the case $|u| = |v|$. In that case, $|u|^2 = |v|^2 = 10$. So we have 8 choices for u : either $\pm 1 \pm 3i$ or $\pm 3 \pm i$. Each such choice determines v , namely $v = 10/u = \bar{u}$. So we have 8 pairs in the case $|u| = |v|$.

Finally, we have the case $|u| > |v|$. By symmetry, it has the same count as the first case $|u| < |v|$. So we have 20 pairs in this case.

Altogether, the number of pairs is $20 + 8 + 20$, which is 48.

Problem 12 Let b_1, b_2, b_3, c_1, c_2 , and c_3 be real numbers such that for every real number x , we have

$$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = (x^2 + b_1x + c_1)(x^2 + b_2x + c_2)(x^2 + b_3x + c_3).$$

Compute $b_1c_1 + b_2c_2 + b_3c_3$.

Answer: -1

Solution: Let P be the polynomial defined by

$$P(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1.$$

Note that $(x + 1)P(x) = x^7 + 1$. So the roots of P are on the unit circle. Hence the roots of each quadratic factor $x^2 + b_kx + c_k$ are also on the unit circle. Because each quadratic factor has real coefficients, its roots come in conjugate pairs. Because the roots are on the unit circle, each c_k is 1.

When we expand the product of the three quadratic factors, we get a polynomial of the form

$$x^6 + (b_1 + b_2 + b_3)x^5 + \dots$$

Because the coefficient of x^5 in P is -1 , we see that $b_1 + b_2 + b_3 = -1$. So we have

$$b_1c_1 + b_2c_2 + b_3c_3 = b_1 + b_2 + b_3 = \boxed{-1}.$$

Problem 13 Alice, Beth, Carla, Dana, and Eden play a game in which each of them simultaneously closes her eyes and randomly chooses two of the others to point at (one with each hand). A participant loses if she points at someone who points back at her; otherwise, she wins. Find the probability that all five girls win. Express your answer as a fraction in simplest form.

Answer: $\frac{1}{324}$

Solution: Each of the five participants has $\binom{4}{2} = 6$ choices of where to point. So the total number of choices is 6^5 .

Consider a directed graph whose vertices are the five participants. Draw an edge from one participant to another if the first points to the second. The number of edges is $5 \cdot 2 = 10$. The number of unordered pairs of vertices is $\binom{5}{2} = 10$. In an all-winning game, each unordered pair has at most one edge. Because the number of edges equals the number of unordered pairs, each unordered pair must have exactly one edge. In human terms, that means for each pair of participants, exactly one of them points to the other.

Next, let's count the number of all-winning outcomes. Alice has $\binom{4}{2} = 6$ choices of where to point. For concreteness, suppose that Alice points to Beth and Carla. That means Dana and Eden will point to Alice. Between Beth and Carla, one will point to the other (2 choices); for concreteness, suppose that Beth points to Carla. Between Dana and Eden, one will point to the other (2 choices); for concreteness, suppose that Dana points to Eden. Now there is exactly one way to complete the picture. So the total number of all-winning outcomes is $6 \cdot 2 \cdot 2 = 24$.

Therefore, the probability of an all-winning outcome is $\frac{24}{6^5} = \frac{2^3 \cdot 3}{2^5 \cdot 3^5}$, which simplifies to $\boxed{\frac{1}{324}}$.

Problem 14 We call a set X of real numbers *three-averaging* if for every two distinct elements a and b of X , there exists an element c in X (different from both a and b) such that the number $(a + b + c)/3$ also belongs to X . For

instance, the set $\{0, 1008, 2016\}$ is three-averaging. What is the least possible number of elements in a three-averaging set with more than 3 elements?

Answer: 5

Solution: Every set of 3 real numbers that form an arithmetic sequence is three-averaging. For example, the sets $\{0, 1, 2\}$, $\{1, 2, 3\}$, and $\{0, 3, 6\}$ are three-averaging.

We claim that the 5-element set $S = \{0, 1, 2, 3, 6\}$ is three-averaging. Most pairs (a, b) in the definition of three-averaging follow from the sets $\{0, 1, 2\}$, $\{1, 2, 3\}$, and $\{0, 3, 6\}$ being three-averaging. The only remaining pairs that we need to check are $(1, 6)$ and $(2, 6)$. If (a, b) is $(1, 6)$, then $c = 2$ leads to $(a + b + c)/3 = 3$, which is in the set S . If (a, b) is $(2, 6)$, then $c = 1$ also leads to $(a + b + c)/3 = 3$. So S is indeed three-averaging.

We claim that no 4-element set is three-averaging. Let $X = \{p, q, r, s\}$ be a set of 4 real numbers. Without loss of generality, assume that $p < q < r < s$. For contradiction's sake, assume that X is three-averaging. By the definition of three-averaging (with $a = q$ and $b = r$), either $(p + q + r)/3$ or $(q + r + s)/3$ is in X . Without loss of generality, assume that $(p + q + r)/3$ is in X . Note that $(p + q + r)/3$ is strictly between p and r . So $(p + q + r)/3$ is equal to q . Hence $p, q,$ and r forms an arithmetic sequence. By translating, we may assume that $p = 0$. By scaling, we may assume that $q = 1$. So $r = 2$.

Choose $a = s$. If s is an integer, then choose $b = s \pmod{3}$; otherwise, choose $b = p$. By the definition of three-averaging, there is a c in X (different from a and b) such that $(a + b + c)/3$ is in X . Note that $(a + b + c)/3$ is strictly between p and s . So $(a + b + c)/3$ is q or r . In particular, $(a + b + c)/3$ is an integer. So $a + b + c$ is a multiple of 3. Because b and c are integers, a is an integer too. Working mod 3, we have

$$a + b + c \not\equiv a + b + b \equiv b + b + b \equiv 0 \pmod{3}.$$

(We know that $c \not\equiv b \pmod{3}$ because b and c are different numbers from the set $\{0, 1, 2\}$.) This contradicts $a + b + c$ being a multiple of 3. Hence our assumption that X is three-averaging is false. So no 4-element set is three-averaging.

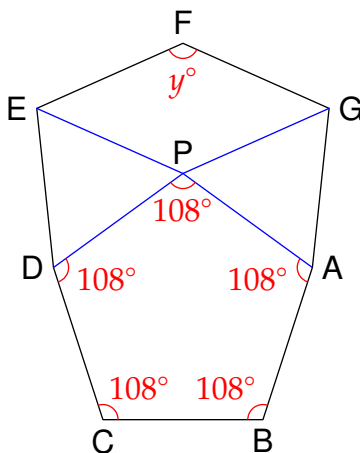
We constructed a 5-element set that is three-averaging and showed that no 4-element set is three-averaging. So the answer to the problem is $\boxed{5}$.

Problem 15 Let H be a convex, equilateral heptagon whose angles measure (in degrees) $168^\circ, 108^\circ, 108^\circ, 168^\circ, x^\circ, y^\circ,$ and z° in clockwise order. Compute

the number y .

Answer: 132

Solution: Label the vertices of the heptagon A, B, C, D, E, F , and G . Draw a regular pentagon (with new vertex P) so that it shares the two 108° angles, as shown below.



Note that $\angle PDE$ is $168^\circ - 108^\circ = 60^\circ$. Because $PD = DE$, triangle PDE is equilateral. By symmetry, triangle PAG is also equilateral. So $PEFG$ is a rhombus. Hence

$$y^\circ = \angle EPG = 360^\circ - 60^\circ - 108^\circ - 60^\circ = 132^\circ.$$

Thus $y = \boxed{132}$.

Problem 16 Let $A < B < C < D$ be positive integers such that every three of them form the side lengths of an obtuse triangle. Compute the least possible value of D .

Answer: 14

Solution: The triangle inequality says that $C < A + B$, $D < A + B$, $D < A + C$, and $D < B + C$. Of these inequalities, the only one that matters is $D < A + B$, because the others follow from it. From obtuseness, we get the inequalities $A^2 + B^2 < C^2$, $A^2 + B^2 < D^2$, $A^2 + C^2 < D^2$, and $B^2 + C^2 < D^2$. Of these inequalities, the only two that matter are $A^2 + B^2 < C^2$ and $B^2 + C^2 < D^2$, because the others follow from them.

So we are looking for positive integers $A < B < C < D$ such that $D < A + B$, $A^2 + B^2 < C^2$, and $B^2 + C^2 < D^2$. Trial and error with small integers and their squares eventually leads to the solution $A = 7$, $B = 8$, $C = 11$, and $D = 14$. In what remains, we will show that every solution satisfies $D \geq 14$.

First, we claim that $B < 2A - 3$. From the obtuseness inequalities $A^2 + B^2 < C^2$ and $B^2 + C^2 < D^2$, we deduce that $A^2 + 2B^2 < D^2$. The triangle inequality $D < A + B$ can be written as $D \leq A + B - 1$. So we have

$$2B^2 < D^2 - A^2 = (D - A)(D + A) \leq (B - 1)(2A + B - 1).$$

Dividing both sides by $B - 1$ gives

$$2A + B - 1 > \frac{2B^2}{B - 1} > 2(B + 1) = 2B + 2.$$

Solving for B gives $B < 2A - 3$, as claimed.

Second, we claim that $B > 5$. We know that $A < B$, which can be written as $A \leq B - 1$. By the previous paragraph, we have

$$B < 2A - 3 \leq 2(B - 1) - 3 = 2B - 5.$$

Solving for B gives $B > 5$, as claimed.

Assume that $B = 6$. Then $A \leq B - 1 = 5$. Also $B < 2A - 3$ implies that $A > 4.5$. So $A = 5$. Because $A^2 + B^2 < C^2$, it follows that $C \geq 8$. Because $B^2 + C^2 < D^2$, it follows that $D \geq 11$. But that contradicts the triangle inequality $D < A + B$.

Assume that $B = 7$. Then $A \leq B - 1 = 6$. Also $B < 2A - 3$ implies that $A > 5$. So $A = 6$. Because $A^2 + B^2 < C^2$, it follows that $C \geq 10$. Because $B^2 + C^2 < D^2$, it follows that $D \geq 13$. But that contradicts the triangle inequality $D < A + B$.

So we know that $B \geq 8$. Then $B < 2A - 3$ implies that $A \geq 6$. Because $A^2 + B^2 < C^2$, it follows that $C \geq 11$. Because $B^2 + C^2 < D^2$, it follows that $D \geq 14$. That's what we wanted to show. So the least possible value of D is $\boxed{14}$.

Problem 17 We define the weight W of a positive integer as follows: $W(1) = 0$, $W(2) = 1$, $W(p) = 1 + W(p + 1)$ for every odd prime p , $W(c) = 1 + W(d)$ for every composite c , where d is the greatest proper factor of c . Compute the greatest possible weight of a positive integer less than 100.

Answer: 12

Solution: Let's compute the first 13 weights.

$$W(n) \begin{array}{c|cccccccccccc} n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ \hline & 0 & 1 & 3 & 2 & 5 & 4 & 4 & 3 & 4 & 6 & 6 & 5 & 6 \end{array}$$

In particular, if $n \leq 13$, then $W(n) \leq 6$.

Suppose that $n \leq 26$. We claim that $W(n) \leq 8$. If $n \leq 13$, then $W(n) \leq 6 < 8$. If n is composite, then its greatest proper factor d is at most $n/2 \leq 13$, so

$$W(n) = 1 + W(d) \leq 1 + 6 < 8.$$

If n is an odd prime, then

$$W(n) = 1 + W(n+1) = 1 + 1 + W\left(\frac{n+1}{2}\right) \leq 1 + 1 + 6 = 8.$$

In every case, $W(n) \leq 8$.

In the same way, we can show that if $n \leq 52$, then $W(n) \leq 10$. Similarly, if $n \leq 104$, then $W(n) \leq 12$.

Let's find a positive integer less than 100 whose weight is 12. It helps to find a nearly-doubling sequence of primes. Note that 19 is prime, $2 \cdot 19 - 1 = 37$ is prime, and $2 \cdot 37 - 1 = 73$ is prime. So

$$W(73) = 2 + W(37) = 4 + W(19) = 6 + W(10) = 6 + 6 = 12.$$

Hence the greatest possible weight of a positive integer less than 100 is $\boxed{12}$.

Problem 18 Let $T = \{1, 2, 3, \dots, 14, 15\}$. Say that a subset S of T is *handy* if the sum of all the elements of S is a multiple of 5. For example, the empty set is handy (because its sum is 0) and T itself is handy (because its sum is 120). Compute the number of handy subsets of T .

Answer: 6560

Solution: We will identify a subset of T with a red-blue coloring of T : color the elements of the subset red and the nonelements blue. Imagine slicing every coloring into three parts of length 5 (namely 1 to 5, 6 to 10, and 11 to 15). In every coloring, either all three parts are monochromatic or not.

Let's deal with the first case: all three parts of the coloring are monochromatic. There are $2^3 = 8$ such colorings. Every such coloring corresponds to a handy subset.

Next, let's deal with the second case: at least one of the three parts of the coloring is not monochromatic. There are $2^{15} - 2^3 = 32,760$ such colorings. Given such a coloring, consider the first part that is not monochromatic. We can apply an arbitrary cyclic shift of this part of the coloring without affecting which parts are monochromatic. So the set of colorings in this case can be partitioned into classes of 5 colorings each. In every class, we claim that exactly one of the 5 colorings corresponds to a handy subset. (Suppose we have a subset R of 1 to 5, or 6 to 10, or 11 to 15. When we shift it cyclically one position to the right, its sum goes up by $|R| \pmod{5}$. So the 5 possible shifts lead to increases mod 5 of 0, $|R|$, $2|R|$, $3|R|$, and $4|R|$. Because $|R|$ is not a multiple of 5, these offsets cover every mod 5 residue exactly once.) Hence the number of colorings in this case that correspond to a handy subset is $(2^{15} - 2^3)/5$, which is 6552.

Combining the two cases, we see that the number of handy subsets is $6552 + 8$, which is 6560.

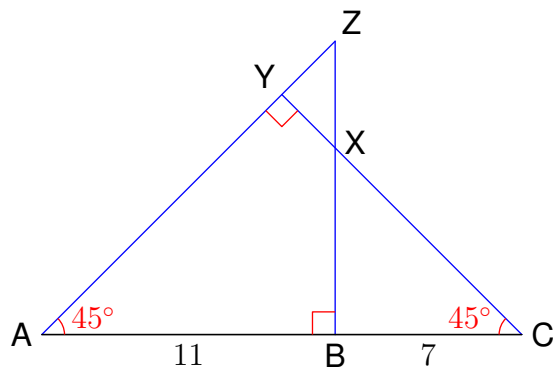
Alternative Solution: We will use a “roots of unity” filter. Let F be the polynomial defined by $F(x) = (1+x)(1+x^2) \cdots (1+x^{15})$. By expanding, we see that $F(x)$ is the sum of $x^{\text{sum}(S)}$ over all subsets S of T , where $\text{sum}(S)$ is the sum of the elements of S . Let ω be a 5th root of unity (different from 1). Note that if n is an integer, then $1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n}$ is equal to 5 when n is a multiple of 5, and is equal to 0 otherwise. So the number of handy subsets is $[F(1) + F(\omega) + F(\omega^2) + F(\omega^3) + F(\omega^4)]/5$. We see that $F(1) = 2^{15}$. Let G be the polynomial defined by $G(y) = (y + \omega^0)(y + \omega^1)(y + \omega^2)(y + \omega^3)(y + \omega^4)$. Its roots are the 5th roots of -1 . So $G(y) = y^5 + 1$. We also see that $F(\omega) = G(1)^3 = 2^3 = 8$. Similarly, $F(\omega^2) = F(\omega^3) = F(\omega^4) = 8$. Hence the number of handy subsets is $(2^{15} + 4 \cdot 2^3)/5$, which is 6560.

Problem 19 In the coordinate plane, consider points $A = (0, 0)$, $B = (11, 0)$, and $C = (18, 0)$. Line ℓ_A has slope 1 and passes through A . Line ℓ_B is vertical and passes through B . Line ℓ_C has slope -1 and passes through C . The three lines ℓ_A , ℓ_B , and ℓ_C begin rotating clockwise about points A , B , and C , respectively. They rotate at the same angular rate. At any given time, the three lines form a triangle. Determine the largest possible area of such a triangle.

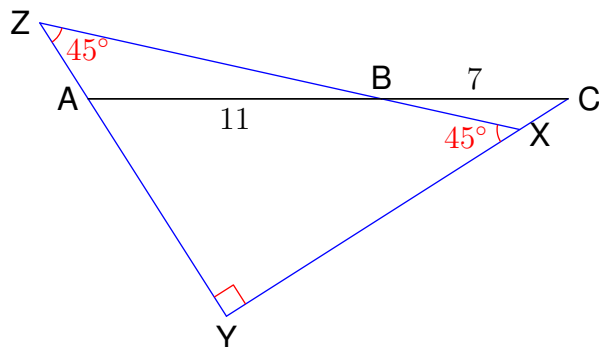
Answer: 85

Solution: Let X be the point where lines ℓ_B and ℓ_C intersect, let Y be the point where lines ℓ_A and ℓ_C intersect, and let Z be the point where lines ℓ_A

and ℓ_B intersect. Here is a picture of the starting position.



Note that $\triangle XYZ$ starts off as a 45-45-90 triangle, with a right angle at Y . Because the three lines rotate at the same rate, the angles they form with each other will remain the same. So $\triangle XYZ$ will remain a 45-45-90 triangle, with a right angle at Y . Here is a typical picture after rotation.



Other cases for the picture are possible too; we will handle them together. Let $\alpha = \angle CAY$, an angle between 0° and 90° . So $\angle BAY = \alpha$. Hence $\angle BAZ$ is either α or $180^\circ - \alpha$, depending on the case. Note that $\angle BZA$ is either 45° or 135° . Applying the Law of Sines to $\triangle BAZ$, we see that $BZ = 11\sqrt{2} \sin \alpha$. Note that $\angle ACY$ is $90^\circ - \alpha$. So $\angle BCY$ is $90^\circ - \alpha$. Hence $\angle BCX$ is either $90^\circ - \alpha$ or $90^\circ + \alpha$. Note that $\angle BXC$ is either 45° or 135° . Applying the Law of Sines to $\triangle BCX$, we see that $XB = 7\sqrt{2} \cos \alpha$. Hence XZ is either $7\sqrt{2} \cos \alpha + 11\sqrt{2} \sin \alpha$ or $|7\sqrt{2} \cos \alpha - 11\sqrt{2} \sin \alpha|$, depending on whether B is between X and Z or not. By the Cauchy-Schwarz inequality, we have

$$XZ \leq \left[(7\sqrt{2})^2 + (11\sqrt{2})^2 \right]^{1/2} = \sqrt{340}.$$

Equality is achieved when α is the acute angle such that $7 \sin \alpha = 11 \cos \alpha$, with B between X and Z , as in the diagram above. So the largest possible value of XZ is $\sqrt{340}$.

Note that XZ is the hypotenuse of $\triangle XYZ$. The leg XY of $\triangle XYZ$ is $XZ/\sqrt{2}$. So the area of $\triangle XYZ$ is $(XZ)^2/4$. Hence the largest possible area of $\triangle XYZ$ is $340/4$, which is 85.

Problem 20 Let a_1, a_2, a_3, a_4 , and a_5 be random integers chosen independently and uniformly from the set $\{0, 1, 2, \dots, 23\}$. (Note that the integers are not necessarily distinct.) Find the probability that

$$\sum_{k=1}^5 \operatorname{cis}\left(\frac{a_k \pi}{12}\right) = 0.$$

(Here $\operatorname{cis} \theta$ means $\cos \theta + i \sin \theta$.) Express your answer as a fraction in simplest form.

Answer: $\frac{35}{27,648}$

Solution: Let $\omega = \operatorname{cis} \frac{\pi}{12}$, a 24th root of unity. Note that $\operatorname{cis} \frac{a_k \pi}{12}$ is ω^{a_k} . So we're looking for 5 powers of ω (not necessarily distinct) that have a vanishing sum (add up to zero).

When does a sum of powers of ω vanish? One example is $\omega^0 + \omega^{12}$, or any rotation of it such as $\omega^5 + \omega^{17}$. We'll call such an example a diameter, because the segment with endpoints ω^5 and ω^{17} in the complex plane is a diameter of the unit circle. Another example is $\omega^0 + \omega^8 + \omega^{16}$, or any rotation of it such as $\omega^7 + \omega^{15} + \omega^{23}$. We'll call such an example an equilateral triangle, because the triangle with vertices ω^7, ω^{15} , and ω^{23} is equilateral. A sum of diameters and equilateral triangles, such as $(\omega^5 + \omega^{17}) + (\omega^7 + \omega^{15} + \omega^{23})$ or $(\omega^0 + \omega^{12}) + (\omega^0 + \omega^8 + \omega^{16})$, also vanishes. We claim that the converse is true: every sum of powers of ω that vanishes can be partitioned into diameters and equilateral triangles. We will prove this claim later. For now, we shall assume that it is true.

We are looking at tuples $(a_1, a_2, a_3, a_4, a_5)$ such that each a_k is in the set $\{0, 1, 2, \dots, 23\}$. The number of such 5-tuples is 24^5 . We would like to count the number of such tuples for which the sum of the ω^{a_k} vanishes. By the claim above, the ω^{a_k} must split into an equilateral triangle and a diameter. We have 8 choices for the equilateral triangle. For now, fix the choice of the equilateral triangle.

The diameter either shares an endpoint with the equilateral triangle or is disjoint from the triangle. In the first case (sharing), there are 3 choices for the diameter. There are $5!/2! = 60$ ways to choose the a_k from the triangle and diameter. So the first case has $3 \cdot 60 = 180$ choices. In the second case (disjoint), there are $12 - 3 = 9$ choices for the diameter. There are $5! = 120$ ways to choose the a_k from the triangle and diameter. So the second case has $9 \cdot 120 = 1080$ choices. Combining the two cases gives $180 + 1080 = 1260$ choices. Multiplying by 8 (the number of choices for the equilateral triangle) gives a total of 10,060 tuples with a vanishing sum. Hence the probability of

vanishing is $\frac{10,060}{24^5}$, which simplifies to $\frac{35}{27,648}$.

We still need to prove the claim: every sum of powers of ω that vanishes can be partitioned into diameters and equilateral triangles. We will use some facts about polynomials. Because $x^{12} + 1$ factors into $x^4 + 1$ times $x^8 - x^4 + 1$, it follows that the polynomial $x^8 - x^4 + 1$ has ω as a root. Conversely, it is known that every polynomial with rational coefficients having ω as a root is divisible by the 24th cyclotomic polynomial Φ , which is $\Phi(x) = x^8 - x^4 + 1$. Note that every exponent of Φ is a multiple of 4.

Suppose that a sum of powers of ω vanishes. Let P be the polynomial that is this sum of powers. It is of the form $b_0 + b_1x + \dots + b_{23}x^{23}$, where the b_j are nonnegative integers. It has ω as a root. Let Q be the polynomial $c_0 + c_1x + \dots + c_{11}x^{11}$, where $c_j = b_j - b_{j+12}$. Because $\omega^{12} = -1$, it follows that Q also has ω as a root. Hence Q is divisible by Φ ; say $Q = R \cdot \Phi$. Split Q into four parts according to the exponent mod 4: $Q_0(x) = c_0 + c_4x^4 + c_8x^8$, $Q_1(x) = c_1x + c_5x^5 + c_9x^9$, $Q_2(x) = c_2x^2 + c_6x^6 + c_{10}x^{10}$, and $Q_3(x) = c_3x^3 + c_7x^7 + c_{11}x^{11}$. Because every exponent of Φ is a multiple of 4 and $Q = R \cdot \Phi$, it follows that Q_0, Q_1, Q_2 , and Q_3 are each divisible by Φ . (Namely, split R into four parts R_0, R_1, R_2 , and R_3 according to the exponent mod 4. Then $Q_0 = R_0 \cdot \Phi$ and so on.)

Let's look at Q_0 first. Because Q_0 is divisible by Φ , it follows that $c_0 = -c_4 = c_8$. In other words, $b_0 - b_{12} = b_8 - b_{20} = b_{16} - b_4$. Let's call their common value t . If $t = 0$, then $b_0 = b_{12}$, $b_8 = b_{20}$, and $b_4 = b_{16}$; in other words, these terms split into diameters. (Namely, b_0 copies of $\omega^0 + \omega^{12}$, b_8 copies of $\omega^8 + \omega^{20}$, and b_{16} copies of $\omega^4 + \omega^{16}$.) If $t > 0$, then after removing diameters (namely b_{12} copies of $\omega^0 + \omega^{12}$, b_{20} copies of $\omega^8 + \omega^{20}$, and b_4 copies of $\omega^4 + \omega^{16}$), we have t copies of the equilateral triangle $\omega^0 + \omega^8 + \omega^{16}$. If $t < 0$, then after removing diameters, we have $-t$ copies of the equilateral

triangle $\omega^4 + \omega^{12} + \omega^{20}$. A similar analysis applies to Q_1 , Q_2 , and Q_3 . Hence we have proved the claim that a vanishing sum of powers of ω splits into diameters and equilateral triangles.

Note: Our claim that a vanishing sum of 24th roots of unity splits into vanishing diameters and triangles also follows from a theorem of N. G. de Bruijn (1953) in his paper “On the Factorization of Cyclic Groups”. His Theorem 2 proves a more general result, with 24 replaced by a product of two prime powers.

Credits

The authors of the 20 problems are as follows.

Ravi Boppana: 9

Mathew Crawford: 13, 20

Ken Fan: 15, 19

Oleg Kryzhanovsky: 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 14, 16, 17, 18