The Advantage Testing Foundation
Math Prize for Girls Olympiad

Thursday, November 16, 2017

Time Limit: 4 hours

1. Given positive integers n and k, say that n is k*-solvable* if there are positive integers a_1, a_2, ..., a_k (not necessarily distinct) such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1$$

and

$$a_1 + a_2 + \cdots + a_k = n.$$

Prove that if n is k-solvable, then $42n + 12$ is $(k + 3)$-solvable.

2. Let n be a positive integer. Prove that there exist polynomials P and Q with real coefficients such that for every real number x, we have $P(x) \geq 0$, $Q(x) \geq 0$, and

$$1 - x^n = (1 - x)P(x) + (1 + x)Q(x).$$

3. Let $ABCD$ be a cyclic quadrilateral such that $\angle BAD \leq \angle ADC$. Prove that $AC + CD \leq AB + BD$.

4. A *lattice point* is a point in the plane whose two coordinates are both integers. A *lattice line* is a line in the plane that contains at least two lattice points. Is it possible to color every lattice point red or blue such that every lattice line contains exactly 2017 red lattice points? Prove that your answer is correct.