The Advantage Testing Foundation

Math Prize for Girls at MIT

2017 Olympiad Solutions

Problem 1 Given positive integers n and k, say that n is k-*solvable* if there are positive integers a_1, a_2, $\ldots$, a_k (not necessarily distinct) such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1$$

and

$$a_1 + a_2 + \cdots + a_k = n.$$

Prove that if n is k-solvable, then $42n + 12$ is $(k + 3)$-solvable.

Solution: Suppose that n is k-solvable. Then there are positive integers a_1, a_2, $\ldots$, a_k such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1$$

and

$$a_1 + a_2 + \cdots + a_k = n.$$

Let b_1, b_2, $\ldots$, b_k, b_{k+1}, b_{k+2}, b_{k+3} be the positive integers defined by $b_i = 42a_i$ (for $1 \le i \le k$), $b_{k+1} = 2$, $b_{k+2} = 3$, and $b_{k+3} = 7$. Note that

$$\begin{aligned}
\sum_{i=1}^{k+3} \frac{1}{b_i} &= \sum_{i=1}^{k} \frac{1}{b_i} + \frac{1}{b_{k+1}} + \frac{1}{b_{k+2}} + \frac{1}{b_{k+3}} \\
&= \sum_{i=1}^{k} \frac{1}{42a_i} + \frac{1}{2} + \frac{1}{3} + \frac{1}{7} \\
&= \frac{1}{42} \sum_{i=1}^{k} \frac{1}{a_i} + \frac{41}{42} \\
&= \frac{1}{42} \cdot 1 + \frac{41}{42} \\
&= 1.
\end{aligned}$$

Also

$$\begin{aligned}\sum_{i=1}^{k+3} b_i &= \sum_{i=1}^{k} b_i + b_{k+1} + b_{k+2} + b_{k+3} \\ &= \sum_{i=1}^{k} 42a_i + 2 + 3 + 7 \\ &= 42\sum_{i=1}^{k} a_i + 12 \\ &= 42n + 12.\end{aligned}$$

Hence $42n + 12$ is $(k+3)$-solvable.

Problem 2 Let n be a positive integer. Prove that there exist polynomials P and Q with real coefficients such that for every real number x, we have $P(x) \geq 0$, $Q(x) \geq 0$, and

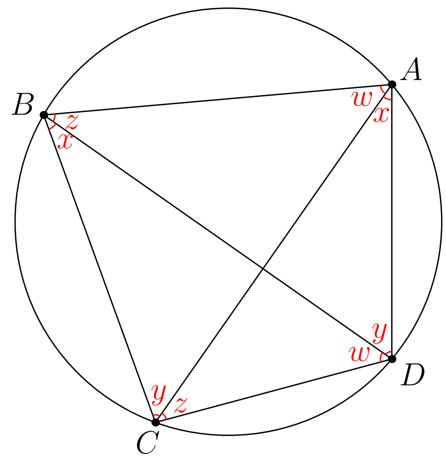
$$1 - x^n = (1 - x)P(x) + (1 + x)Q(x).$$

Solution: First, suppose that n is odd. Let $P(x) = \sum_{k=0}^{n-1} x^k$. Let $Q(x) = 0$. Because $P(x)$ is a geometric series, we have $(1 - x)P(x) = 1 - x^n$. Hence $1 - x^n = (1 - x)P(x) + (1 + x)Q(x)$. We claim that P is nonnegative. If $x \geq 0$, then every term of $P(x)$ is nonnegative, so $P(x)$ itself is nonnegative. If $x < 1$, then both $1 - x$ and $1 - x^n$ are positive, so $P(x)$ is positive. Either way, $P(x)$ is nonnegative. And of course Q, the all-zero polynomial, is nonnegative.

Next, suppose that n is even. Define $R(x) = \frac{1}{2}\sum_{k=0}^{n/2-1} x^{2k}$. Let $P(x) = (1 + x)^2 R(x)$. Let $Q(x) = (1 - x)^2 R(x)$. Because $R(x)$ is a geometric series, we have $2(1 - x^2)R(x) = 1 - x^n$. Note that $(1 - x)(1 + x)^2 + (1 + x)(1 - x)^2$ is $2(1 - x^2)$. Multiplying by $R(x)$, we see that $(1 - x)P(x) + (1 + x)Q(x)$ is $1 - x^n$. Because every term of R is a square, R is nonnegative. Hence P and Q are nonnegative.

Problem 3 Let $ABCD$ be a cyclic quadrilateral such that $\angle BAD \leq \angle ADC$. Prove that $AC + CD \leq AB + BD$.

Solution: Because $ABCD$ is cyclic, its vertices lie on a circle. By the inscribed angle theorem, $\angle BAC = \angle BDC$; let w be their common measure. Similarly, let $x = \angle CAD = \angle CBD$, let $y = \angle ACB = \angle ADB$, and let $z = \angle ABD = \angle ACD$. See the following picture.



As the picture shows, $\angle BAD = w + x$ and $\angle ADC = w + y$. Because $\angle BAD \le \angle ADC$, we have $x \le y$. Because the angles in a triangle add up to 180°, the sum $w + x + y + z$ is 180°. In particular, $x + y + z$ is at most 180°.

By scaling, we may assume that the diameter of the circle is 1. By the extended law of sines, we have $AC = \sin(x + z)$. Similarly, $CD = \sin x$, $AB = \sin y$, and $BD = \sin(y + z)$.

We will use the sum-to-product identity

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\,.$$

In particular, we have

$$AC + CD = \sin(x + z) + \sin x = 2\sin\Bigl(x + \frac{z}{2}\Bigr)\cos\frac{z}{2}\,.$$

Similarly, we have

$$AB + BD = \sin y + \sin(y + z) = 2\sin\Bigl(y + \frac{z}{2}\Bigr)\cos\frac{z}{2}\,.$$

Because $x \le y$ and $x + y + z \le 180^\circ$, we have

$$0^\circ \le x + \frac{z}{2} \le y + \frac{z}{2} \le 180^\circ - \Bigl(x + \frac{z}{2}\Bigr) \le 180^\circ.$$

Hence $\sin\bigl(x + \frac{z}{2}\bigr) \le \sin\bigl(y + \frac{z}{2}\bigr)$. Because $0^\circ \le z \le 180^\circ$, we have $\cos\frac{z}{2} \ge 0$. Therefore

$$AC + CD = 2\sin\Bigl(x + \frac{z}{2}\Bigr)\cos\frac{z}{2} \le 2\sin\Bigl(y + \frac{z}{2}\Bigr)\cos\frac{z}{2} = AB + BD.$$

Problem 4 A *lattice point* is a point in the plane whose two coordinates are both integers. A *lattice line* is a line in the plane that contains at least two lattice points. Is it possible to color every lattice point red or blue such that every lattice line contains exactly 2017 red lattice points? Prove that your answer is correct.

Solution: Yes, there is a red-blue coloring such that every lattice line contains exactly 2017 red lattice points. We will construct such a coloring below.

First, note that a line containing two lattice points must contain infinitely many lattice points. So every lattice line contains infinitely many lattice points.

The set of integers is countable. So the set of lattice points is countable. Hence the set of pairs of lattice points is countable. Therefore, because two distinct points determine a line, the set of lattice lines is countable. Let ℓ_1, ℓ_2, ... be an enumeration of all distinct lattice lines.

We will start with every lattice point colored blue. At every time step (1, 2, ...), we will recolor a finite number of the lattice points red. We will maintain the invariant that, just before every time t (a positive integer), the lines $\ell_1, \ell_2, \ldots, \ell_{t-1}$ each contain exactly 2017 red points, while each of the other lines contain at most 2017 red points.

Suppose we are at time t. The set of (current) red points is finite. So the set of pairs of red points is finite. Hence the set of lines with at least two red points is finite. Therefore the set of lines with exactly 2017 red points is finite. Let S be the finite set of lines with exactly 2017 red points, excluding line ℓ_t. Every line in S intersects ℓ_t in at most one point. So the set of lattice points on ℓ_t that are also on some line in S is finite; let P be this finite set of points. Because P is finite, and at most 2017 points on ℓ_t are red (by the invariant), there are infinitely many lattice points on ℓ_t that are neither in P nor currently red. Recolor up to 2017 of these points red until ℓ_t has exactly 2017 red points.

We claim that the invariant from two paragraphs ago is true for every positive integer t. We will prove the claim by induction. The base case $t = 1$ holds because we start with the all-blue coloring. Suppose the invariant is true for t; we will prove it for $t + 1$. Because every line except ℓ_t itself intersects ℓ_t in at most one point, every line except ℓ_t will gain at most one red point during time t. So every line with fewer than 2017 red points just before time t will have at most 2017 red points just after time t. By construction, every line with exactly 2017 red points just before time t will not gain a red point during time t. In particular, the lines ℓ_1, ℓ_2, …, ℓ_{t-1} will still each contain exactly 2017 red points. By construction, ℓ_t itself will also have exactly 2017 red points just after time t. Hence we have proved the claim by induction.

Consider the red-blue coloring that is the limit of the colorings above as the time t approaches infinity. By the invariant, every lattice line contains exactly 2017 red lattice points in the limit. Hence the limit coloring is the desired coloring.

Note: This problem was proposed by Oleg Kryzhanovsky.