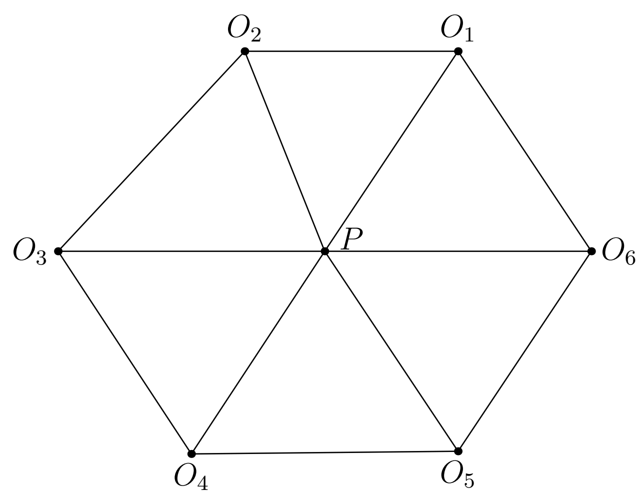
The Advantage Testing Foundation

Math Prize for Girls at MIT

2018 Olympiad Solutions

Problem 1 Let P be a point in the plane. Suppose that P is inside (or on) each of 6 circles ω_1, ω_2, $\ldots$, ω_6 in the plane. Prove that there exist distinct i and j so that the center of circle ω_i is inside (or on) circle ω_j.

Solution: Let the centers of circles ω_1, ω_2, $\ldots$, ω_6 be called O_1, O_2, $\ldots$, O_6 respectively. Without loss of generality, assume that O_1, O_2, $\ldots$, O_6 are arranged in counterclockwise order with respect to P.



The angles $\angle O_1PO_2$, $\angle O_2PO_3$, $\ldots$, $\angle O_6PO_1$ add up to $360°$. By the pigeonhole principle, at least one of the six angles must be less than or equal to $60°$. Without loss of generality, assume that $\angle O_1PO_2 \le 60°$.

Consider the triangle $\triangle O_1PO_2$. Because $\angle O_1PO_2 \le 60°$, either $\angle O_1O_2P \ge 60°$ or $\angle O_2O_1P \ge 60°$. Without loss of generality, assume that $\angle O_1O_2P \ge 60°$. In particular, $\angle O_1O_2P \ge \angle O_1PO_2$.

It is a well-known fact that in every triangle, the side lengths are sorted in the same order as their opposite angles. Using this fact, we conclude that $O_1P \ge O_1O_2$.

Let the radius of circle ω_1 be r_1. Since P is in or on ω_1, we have $O_1P \le r_1$. It follows that $O_1O_2 \le r_1$. Thus O_2 is in or on ω_1.

Problem 2 Let $d(n)$ be the number of positive divisors of a positive integer n. Let $\mathbb{N}$ be the set of all positive integers. Say that a function F from $\mathbb{N}$ to $\mathbb{N}$ is *divisor-respecting* if $d(F(mn)) = d(F(m))d(F(n))$ for all positive integers m and n, and $d(F(n)) \le d(n)$ for all positive integers n. Find all divisor-respecting functions. Justify your answer.

Solution: The identically-1 function, namely $F(n) = 1$ for every positive integer n, is divisor-respecting, because $d(1) = 1$. We claim that the only divisor-respecting function is the identically-1 function. Let F be a divisor-respecting function.

First we prove a lemma about prime numbers.

Lemma *For every prime number p, we have $F(p) = 1$.*

PROOF Applying the definition of divisor-respecting to $m = n = p$, we get that

$$d(F(p^2)) = d(F(p))^2$$

and

$$d(F(p^2)) \le d(p^2).$$

Note that $d(p^2) = 3$, as the divisors of p^2 are 1, p, and p^2. Thus

$$d(F(p))^2 = d(F(p^2)) \le d(p^2) = 3.$$

Since $d(F(p))$ is a positive integer, we deduce that $d(F(p)) = 1$. Finally, we note that the only positive integer with one divisor is 1, so $F(p) = 1$. ∎

We now prove that $F(n) = 1$ for every positive integer n. Suppose the prime factorization of n is $n = p_1p_2 \cdots p_k$ for some prime numbers p_1, p_2, ..., p_k. By the divisor-respecting property, we can write

$$d(F(n)) = d(F(p_1p_2 \cdots p_k)) = d(F(p_1))d(F(p_2)) \cdots d(F(p_k)).$$

By our lemma, we know that $d(F(p_i)) = 1$ for all $1 \le i \le k$. So $d(F(n)) = 1$. Furthermore, $F(n) = 1$, since the only positive integer with one divisor is 1.

Therefore, F is the identically-1 function.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 3 There is a wooden $3 \times 3 \times 3$ cube and 18 rectangular 3×1 paper strips. Each strip has two dotted lines dividing it into three unit squares. The full surface of the cube is covered with the given strips, flat or bent. Each flat strip is on one face of the cube. Each bent strip (bent at one of its dotted lines) is on two adjacent faces of the cube. What is the greatest possible number of bent strips? Justify your answer.

Solution: We claim that the greatest possible number of bent strips is 14.

First, we show that placing more than 14 bent strips is impossible. Consider the 8 unit cubes at each corner of the $3 \times 3 \times 3$ cube, each of which has three exposed faces (squares) that must be covered. Note that if a bent strip covers a square on a corner, then it must cover a second square on the same corner. Thus, because bent strips cover corner squares in pairs, at least one of the 3 squares per corner must be covered by a flat strip. Moreover, we note that each flat strip can cover squares on at most two distinct corners. Therefore, we require at least 4 flat strips to cover all 8 corners. So of the 18 total strips, at most 14 can be bent.

To show that 14 bent strips are possible, we construct an example. The example below is drawn on the net of a cube, where each variable represents a separate strip. Flat strips are drawn in red with capital letters.

			i	j	j			
			f	f	e			
			k	k	l			
i	f	k	P	P	P	l	e	j
i	a	a	a	b	m	l	e	d
O	O	O	g	b	m	Q	Q	Q
			g	b	m			
			g	c	n			
			h	c	n			
			h	c	n			
			h	d	d			
			R	R	R			

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 4 For all integers x and y, let $a_{x,y}$ be a real number. Suppose that $a_{0,0} = 0$. Suppose that only a finite number of the $a_{x,y}$ are nonzero. Prove that

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} a_{x,y}(a_{x,2x+y} + a_{x+2y,y}) \le \sqrt{3} \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} a_{x,y}^2 .$$

Solution: Let z be a point (x, y). Define the *length* of z to be the usual Euclidean length:

$$|z| = \sqrt{x^2 + y^2} .$$

Let V be the map defined by $V(z) = (x, y + 2x)$. Let H be the map defined by $H(z) = (x + 2y, y)$. We call these maps V and H because V is a vertical shear and H is a horizontal shear. The inverse of V is the vertical shear given by $V^{-1}(z) = (x, y - 2x)$. The inverse of H is the horizontal shear given by $H^{-1}(z) = (x-2y, y)$. We call $V(z)$, $H(z)$, $V^{-1}(z)$, and $H^{-1}(z)$ the *neighbors* of z.

First we prove a lemma that, roughly speaking, says that most neighbors of a point are farther away from the origin than the original point is.

Lemma *Let z be a point different from $(0, 0)$. Then either (a) three neighbors of z are longer than z and the other neighbor is shorter than z, or (b) two neighbors of z are longer than z and the other two neighbors are the same length as z.*

PROOF Suppose the coordinates of z are (x, y). By replacing x and y with their negations as necessary, we may assume that $x \ge 0$ and $y \ge 0$. By swapping x and y if necessary, we may assume that $x \ge y$. Because z is not $(0, 0)$, we have $x > 0$.

If $y = 0$, then $V(z)$ and $V^{-1}(z)$ are longer than z while $H(z)$ and $H^{-1}(z)$ are the same length as z. So in this case, conclusion (b) holds.

If $x = y$, then $V(z)$ and $H(z)$ are longer than z while $V^{-1}(z)$ and $H^{-1}(z)$ are the same length as z. So again in this case, conclusion (b) holds.

Otherwise $x > y > 0$. Then $V(z)$, $V^{-1}(z)$, and $H(z)$ are longer than z while $H^{-1}(z)$ is shorter than z. So in this remaining case, conclusion (a) holds. ∎

Next we turn to proving inequalities. Let r and s be real numbers. The inequality

$$rs \le \frac{1}{2}r^2 + \frac{1}{2}s^2$$

is trivial. Replacing r with $r/3^{1/4}$ and s with $3^{1/4}s$, we obtain the inequality

$$rs \le \frac{1}{2\sqrt{3}}r^2 + \frac{\sqrt{3}}{2}s^2.$$

In the problem statement, the left-hand side of the inequality is a sum of terms of the form $a_z a_w$, where z and w are neighboring points. Because $a_{0,0} = 0$, we may assume that neither z nor w is $(0,0)$. If $|z| < |w|$, then we will bound the term $a_z a_w$ by

$$a_z a_w \le \frac{1}{2\sqrt{3}}a_z^2 + \frac{\sqrt{3}}{2}a_w^2\,.$$

If $|z| > |w|$, then we will bound the term by

$$a_z a_w \le \frac{\sqrt{3}}{2}a_z^2 + \frac{1}{2\sqrt{3}}a_w^2\,.$$

If $|z| = |w|$, then we will bound the term by

$$a_z a_w \le \frac{1}{2}a_z^2 + \frac{1}{2}a_w^2\,.$$

Bounding the left-hand side of the original inequality as above will make the upper bound a sum of terms of the form a constant times a_z^2.

For a given $z \ne (0,0)$, let's enumerate the terms with a_z^2. By the lemma, either conclusion (a) or conclusion (b) holds. If conclusion (a) holds, then the term $\frac{1}{2\sqrt{3}}a_z^2$ will appear three times and the term $\frac{\sqrt{3}}{2}a_z^2$ once, for a total of $\sqrt{3}\,a_z^2$. If conclusion (b) holds, then the term $\frac{1}{2\sqrt{3}}a_z^2$ will appear twice and the term $\frac{1}{2}a_z^2$ twice, for a total of $(1 + \frac{1}{\sqrt{3}})a_z^2$. Either way, the upper bound is at most $\sqrt{3}\,a_z^2$.

Summing over all z gives the desired inequality.