The Advantage Testing Foundation

Math Prize for Girls at MIT

2019 Olympiad Solutions

Problem 1 Let A_1, A_2, ..., A_n be finite sets. Prove that

$$\left| \bigcup_{1 \le i \le n} A_i \right| \ge \frac{1}{2} \sum_{1 \le i \le n} |A_i| - \frac{1}{6} \sum_{1 \le i < j \le n} |A_i \cap A_j| \, .$$

Recall that if S is a finite set, then its cardinality $|S|$ is the number of elements of S.

Solution: Let S be the union of all the A_i. Given an element x of S, define its *degree* $\deg(x)$ to be the number of i such that $x \in A_i$. We can express $\sum_i |A_i|$ using degrees as follows:

$$\sum_i |A_i| = |\{(i, x) : x \in A_i\}| = \sum_{x \in S} \deg(x).$$

Similarly, we can express $\sum_{i<j} |A_i \cap A_j|$ as follows:

$$\sum_{i<j} |A_i \cap A_j| = |\{(i, j, x) : i < j \text{ and } x \in A_i \text{ and } x \in A_j\}| = \sum_{x \in S} \binom{\deg(x)}{2} \, .$$

To continue, we will need a simple inequality involving a single integer.

Lemma For every integer d, we have

$$1 \ge \frac{1}{2} d - \frac{1}{6} \binom{d}{2} \, .$$

Proof Either $d \le 3$ or $d \ge 4$, so $(d - 3)(d - 4) \ge 0$. Rearranging gives $12 \ge 6d - d(d - 1)$. Dividing by 12 gives the desired inequality. ∎

Given an element x of S, applying the lemma with $d = \deg(x)$ gives

$$1 \geq \frac{1}{2}\deg(x) - \frac{1}{6}\binom{\deg(x)}{2}.$$

Summing over all x gives

$$\sum_{x \in S} 1 \geq \frac{1}{2}\sum_{x \in S}\deg(x) - \frac{1}{6}\sum_{x \in S}\binom{\deg(x)}{2}.$$

The first sum is $|S| = \left|\bigcup_i A_i\right|$. The second sum is $\sum_i |A_i|$. The third sum is $\sum_{i<j}|A_i \cap A_j|$. Hence we are done.

Problem 2 Let ABC be an equilateral triangle with side length 1. Say that a point X on side $\overline{BC}$ is *balanced* if there exists a point Y on side $\overline{AC}$ and a point Z on side $\overline{AB}$ such that the triangle XYZ is a right isosceles triangle with $XY = XZ$. Find with proof the length of the set of all balanced points on side $\overline{BC}$.

Solution: We will show that the the set of balanced points has length $2-\sqrt{3}$. In particular, we will show that the set of balanced points is the interval of points X on side $\overline{BC}$ such that $\frac{\sqrt{3}-1}{2} \leq BX \leq \frac{3-\sqrt{3}}{2}$.

We will solve the problem with complex numbers. To see why complex numbers are useful here, note that a triangle XYZ in the complex plane is a right triangle in counterclockwise order with $XY = YZ$ if and only if $Z - X = (Y - X)i$. That's because the equation $Z - X = (Y - X)i$ says that the vector $\overrightarrow{XZ}$ is a (counterclockwise) rotation by 90° of the vector $\overrightarrow{XY}$. Rearranging $Z - X = (Y - X)i$ gives $(1 - i)X = Z - Yi$.

Because ABC is an equilateral triangle with side length 1, we can place it in the complex plane so that $A = 0$, $B = 1$, and $C = \operatorname{cis} 60^\circ$. In our problem, X is on side $\overline{BC}$, Y is on side $\overline{AC}$, and Z is on side $\overline{AB}$. So triangle XYZ is in counterclockwise order.

Let $X = p + qi$ be an arbitrary point in the complex plane, where p and q are real. We will show that there exists a unique pair of points Y on the line $\overleftrightarrow{AC}$ and Z on the line $\overleftrightarrow{AB}$ such that $(1 - i)X = Z - Yi$. Because Y is on line $\overleftrightarrow{AC}$, it is of the form $rC = r \operatorname{cis} 60^\circ$, where r is real. Because Z is on line $\overleftrightarrow{AB}$, it is of the form $sB = s$, where s is real. The left-hand side of the equation $(1 - i)X = Z - Yi$ is

$$(1 - i)X = (1 - i)(p + qi) = p + q + (q - p)i.$$

The right-hand side of the equation is

$$Z - Yi = s - r \operatorname{cis} 60^\circ = \frac{\sqrt{3}}{2} r + s - \frac{1}{2} ri.$$

Comparing real parts and imaginary parts gives the two equations

$$\begin{aligned} \frac{\sqrt{3}}{2} r + s &= p + q \\ -\frac{1}{2} r &= q - p. \end{aligned}$$

Solving this system of equations gives the unique solution $r = 2p - 2q$ and $s = (1 - \sqrt{3})p + (1 + \sqrt{3})q$.

In our problem, X is on side $\overline{BC}$. In other words, X is of the form $(1-t)B + tC = 1 - t + t \operatorname{cis} 60^\circ$, where $t = BX$. So $X = p + qi$, where $p = 1 - \frac{1}{2}t$ and $q = \frac{\sqrt{3}}{2}t$. By the previous paragraph, we have $r = 2 - (1 + \sqrt{3})t$ and $s = 1 - \sqrt{3} + (1 + \sqrt{3})t$. For Y to be on side $\overline{AC}$ is equivalent to $0 \le r \le 1$, which means $\frac{\sqrt{3}-1}{2} \le t \le \sqrt{3} - 1$. For Z to be on side $\overline{AB}$ is equivalent to $0 \le s \le 1$, which means $2 - \sqrt{3} \le t \le \frac{3-\sqrt{3}}{2}$. Hence Y on $\overline{AC}$ and Z on $\overline{AB}$ is equivalent to $\frac{\sqrt{3}-1}{2} \le t \le \frac{3-\sqrt{3}}{2}$. In other words, X is balanced if and only if $\frac{\sqrt{3}-1}{2} \le BX \le \frac{3-\sqrt{3}}{2}$.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 3 Say that a positive integer is *red* if it is of the form n^{2020}, where n is a positive integer. Say that a positive integer is *blue* if it is not red and is of the form n^{2019}, where n is a positive integer. True or false: Between every two different red positive integers greater than $10^{100,000,000}$, there are at least 2019 blue positive integers. Prove that your answer is correct.

Solution: The answer is True. Suppose we are given two different red positive integers greater than $10^{100,000,000}$. We will show that there are at least 2019 blue positive integers between them. Without loss of generality, we may assume that the two red integers are consecutive red integers, say n^{2020} and $(n + 1)^{2020}$. Because they are greater than $10^{100,000,000}$, we have $n > 10^{10,000}$.

Lemma If $x > 10^{10,000}$, then $(x + 1)^{2020/2019} > x^{2020/2019} + 2019$.

Proof If $x > 10^{10,000}$, then

$$\begin{aligned}(x+1)^{2020/2019} &= (x+1)(x+1)^{1/2019}\\ &> (x+1)x^{1/2019}\\ &= x^{2020/2019} + x^{1/2019}\\ &> x^{2020/2019} + 2019.\end{aligned}$$ ■

Applying the lemma to $x = n$, we have $(n+1)^{2020/2019} > n^{2020/2019} + 2019$. In other words, the length of the open interval $(n^{2020/2019}, (n+1)^{2020/2019})$ is greater than 2019. So there are at least 2019 integers strictly between $n^{2020/2019}$ and $(n+1)^{2020/2019}$. Taking 2019th powers, we see that there are at least 2019 integers of the form m^{2019} (where m is a positive integer) strictly between n^{2020} and $(n+1)^{2020}$. Being strictly between two consecutive red integers, those at least 2019 integers of the form m^{2019} can't be red, and so they are blue. Hence we are done.

Note: This problem was proposed by Oleg Kryzhanovsky.

Problem 4 Let n be a positive integer. Let d be an integer such that $d \geq n$ and d is a divisor of $\frac{n(n+1)}{2}$. Prove that the set $\{1, 2, \ldots, n\}$ can be partitioned into disjoint subsets such that the sum of the numbers in each subset equals d.

Solution: Given integers a and b, define $a \mathinner{..} b$ to be the set $\{a, a+1, \ldots, b\}$. (If $b < a$, then $a \mathinner{..} b$ is the empty set.) Given integers n, c, and d, define a (n, c, d)-*partition* to be a partition of the set $0 \mathinner{..} n$ into c subsets, each with sum d. The distinction between $0 \mathinner{..} n$ and $1 \mathinner{..} n$ doesn't matter for our problem, because we can delete or insert the 0 without changing any sum.

We will start with three lemmas that show how to construct new partitions from smaller ones.

Lemma 1 Let n, c, and d be integers such that $2c \leq n+1$. If there is a $(n-2c, c, d-2n+2c-1)$ partition, then there is a (n, c, d) partition.

Proof Suppose there is a $(n-2c, c, d-2n+2c-1)$ partition. There is also a partition of the set $n-2c+1 \mathinner{..} n$ into c subsets, each with sum $2n-2c+1$: namely, the c pairs $\{n-2c+1, n\}$, $\{n-2c+2, n-1\}$, …, $\{n-c+1, n-c\}$. Combining each of the c subsets of the first partition with one of the c subsets of the second partition gives the desired (n, c, d) partition. ■

Lemma 2 Let n, c, and d be integers such that d is odd and $n \le d \le 2n+1$. If there is a $(d-n-1, c-n+\frac{d-1}{2}, d)$ partition, then there is a (n, c, d) partition.

PROOF Suppose there is a $(d-n-1, c-n+\frac{d-1}{2}, d)$ partition. There is also a partition of the set $d-n\,..\,n$ into $n-\frac{d-1}{2}$ subsets, each with sum d: namely, the pairs $\{d-n, n\}$, $\{d-n+1, n-1\}$, ..., $\{\frac{d-1}{2}, \frac{d+1}{2}\}$. The union of the two partitions is the desired (n, c, d) partition. ∎

Lemma 3 Let n, c, and d be integers such that d is even and $n \le d \le 2n$. If there is a $(d-n-1, 2c-2n+d-1, \frac{d}{2})$ partition, then there is a (n, c, d) partition.

PROOF Suppose there is a $(d-n-1, 2c-2n+d-1, \frac{d}{2})$ partition. By including the singleton set $\{\frac{d}{2}\}$, we obtain a partition of $0\,..\,d-n-1 \cup \{\frac{d}{2}\}$ into $2c-2n+d$ subsets, each with sum $\frac{d}{2}$. By pairing up these subsets, we obtain a partition of $0\,..\,d-n-1 \cup \{\frac{d}{2}\}$ into $c-n+\frac{d}{2}$ subsets, each with sum d.

There is also a partition of the set $d-n\,..\,n$, excluding $\frac{d}{2}$, into $n-\frac{d}{2}$ subsets, each with sum d: namely, the pairs $\{d-n, n\}$, $\{d-n+1, n-1\}$, ..., $\{\frac{d}{2}-1, \frac{d}{2}+1\}$. The union of this partition and the final partition of the previous paragraph is the desired (n, c, d) partition. ∎

Given a finite set S of integers, define $\text{sum}(S)$ to be the sum of the elements of S. For example, $\text{sum}(1\,..\,n)$ is $\frac{n(n+1)}{2}$.

We will now prove the following theorem, which is a restatement of the original problem.

Theorem Let n, c, and d be integers such that $n > 0$, $d \ge n$, and $cd = \text{sum}(1\,..\,n)$. Then there is a (n, c, d) partition.

PROOF The proof is by strong induction on n.

We will divide the proof into five cases. The first case is $d = n$. The second case is $d = n+1$. The third case is $n+1 < d \le 2n$ and d odd. The fourth case is $n+1 < d \le 2n$ and d even. The fifth case is $d > 2n$.

First, let's handle the case $d = n$. Because $cd = \text{sum}(1\,..\,n)$, we have $c = \frac{n+1}{2}$. There is trivially a $(-1, c, 0)$ partition. Hence, by Lemma 1, there is a (n, c, d) partition.

Second, let's handle the case $d = n+1$. Because $cd = \text{sum}(1\,..\,n)$, we have $c = \frac{n}{2}$. There is trivially a $(0, c, 0)$ partition. Hence, by Lemma 1, there is a (n, c, d) partition.

Third, let's handle the case $n+1 < d \leq 2n$ and d odd. Define $n' = d-n-1$ and $c' = c - n + \frac{d-1}{2}$. Because $d \leq 2n$, we have $n' < n$ and so $d > n > n'$. Because $d > n+1$, we have $n' > 0$. We can check that $c'd = \operatorname{sum}(1\,..\,n')$ as follows:

$$c'd = cd - \left(n - \frac{d-1}{2}\right)d = \operatorname{sum}(1\,..\,n) - \operatorname{sum}(d-n\,..\,n) = \operatorname{sum}(1\,..\,n').$$

By induction, there is a (n', c', d) partition. Hence, by Lemma 2, there is a (n, c, d) partition.

Fourth, let's handle the case $n+1 < d \leq 2n$ and d even. Define $n' = d - n - 1$, $c' = 2c - 2n + d - 1$, and $d' = \frac{d}{2}$. Because $d \leq 2n$, we have $n \geq d' > n'$. As in the previous case, we have $n' > 0$ and $c'd' = \operatorname{sum}(1\,..\,n')$. By induction, there is a (n', c', d') partition. Hence, by Lemma 3, there is a (n, c, d) partition.

Fifth and finally, let's handle the case $d > 2n$. Define $n' = n - 2c$ and $d' = d - 2n + 2c - 1$. Because $c > 0$, we have $n' < n$. Because $d > 2n$ and $cd = \operatorname{sum}(1\,..\,n)$, we have $4c \leq n$, which implies $2c \leq n'$. Thus $n' > 0$. Because $cd = \operatorname{sum}(1\,..\,n)$ and $c(2n - 2c + 1) = \operatorname{sum}(n - 2c + 1\,..\,n)$, we have $cd' = \operatorname{sum}(1\,..\,n')$. Because $2c \leq n'$ and $cd' = \operatorname{sum}(1\,..\,n')$, we have $d' > n'$. By induction, there is a (n', c, d') partition. Hence, by Lemma 1, there is a (n, c, d) partition. ∎