The Advantage Testing Foundation

Math Prize for Girls at MIT

2019 Solutions

Problem 1 In the USA, standard letter-size paper is 8.5 inches wide and 11 inches long. What is the largest integer that cannot be written as a sum of a whole number (possibly zero) of 8.5's and a whole number (possibly zero) of 11's?

Answer: 159

Solution: If $8.5a + 11B$ is an integer, where a and B are whole numbers, then $8.5a$ must be an integer, and therefore a must be even. Let $a = 2A$, where A is a whole number, so that $8.5a + 11B = 17A + 11B$. It is a well-known result of Sylvester[1] that if n and m are relatively prime positive integers, then the largest integer that cannot be expressed in the form $An + Bm$, where A and B are whole numbers, is given by $nm - n - m$. Applying this result in the case $n = 17$ and $m = 11$ yields the answer $17(11) - 17 - 11 = \boxed{159}$.

Problem 2 Let a_1, a_2, …, a_{2019} be a sequence of real numbers. For every five indices i, j, k, ℓ, and m from 1 through 2019, at least two of the numbers a_i, a_j, a_k, a_ℓ, and a_m have the same absolute value. What is the greatest possible number of distinct real numbers in the given sequence?

Answer: 8

Solution: If 5 or more distinct absolute values of numbers exist in the sequence, then we could choose the 5 indices so that each has a different absolute value, contradicting the hypothesis. So there are at most 4 distinct absolute values of numbers in the sequence. By the pigeonhole principle, for any 5 terms in the sequence, some two have the same absolute value. Since at most 2 distinct real numbers can have the same absolute value, the answer is $2 \times 4 = \boxed{8}$.

[1] Look up the "Frobenius coin problem" for more information.

Problem 3 The degree measures of the six interior angles of a convex hexagon form an arithmetic sequence (not necessarily in cyclic order). The common difference of this arithmetic sequence can be any real number in the open interval $(-D, D)$. Compute the greatest possible value of D.

Answer: 24

Solution: Without loss of generality, we may assume that the degree measures of the interior angles are $a + kd$, where $0 \le k \le 5$ is an integer and $d \ge 0$. Since the hexagon is convex, we also know that $a > 0$ and $a+5d < 180$. The degree measures of the interior angles add up to $180(6-2)$, thus $6a + 15d = 4(180)$. Solving for a and substituting the result into the inequality, we find $(4(180) - 15d) + 30d < 6(180)$, or $d < 24$. For such d, $6a = 4(180) - 15d > 360$, hence a is positive. Therefore, the answer is $\boxed{24}$.

Problem 4 A paper equilateral triangle with area 2019 is folded over a line parallel to one of its sides. What is the greatest possible area of the overlap of folded and unfolded parts of the triangle?

Answer: 673

Solution: We make frequent use of the fact that the area of an equilateral triangle of side length s is Es^2, for some positive constant E.

Let the side length of the triangle be s. Let l be the length of the crease.

If $0 < l \le s/2$, then the area of overlap is an equilateral triangle with side length l. The area of the overlap is El^2. For $0 < l \le s/2$, this area is maximized with $l = s/2$ which corresponds to an area of $Es^2/4$.

If $s/2 < l < s$, then the area of the overlap is an isosceles trapezoid that can be thought of as an equilateral triangle of side length l with an equilateral triangular tip removed. So construed, let t be the side length of the equilateral triangular tip. Observe that the nonoverlapping parts of the folded model consist of three equilateral triangles: the tip with side length t and two congruent equilateral triangles whose side lengths we will label x. Then $2x+t = s$ and $x+t = l$. Therefore, $t = s-2x = s-2(l-t)$, from which we find $t = 2l-s$. Hence, the are of the overlap is $E(l^2-t^2) = E(3l-s)(s-l)$. This is a quadratic in l with maximum value attained at $l = 2s/3$, which is a value that satisfies the defining conditions of this case. The area when $l = 2s/3$ is $Es^2/3$, which is larger than $Es^2/4$.

We conclude that the maximum area of overlap is 1/3 the area of the original piece of paper, or $2019/3 = \boxed{673}$.

Problem 5 Two ants sit at the vertex of the parabola $y = x^2$. One starts walking northeast (i.e., upward along the line $y = x$) and the other starts walking northwest (i.e., upward along the line $y = -x$). Each time they reach the parabola again, they swap directions and continue walking. Both ants walk at the same speed. When the ants meet for the eleventh time (including the time at the origin), their paths will enclose 10 squares. What is the total area of these squares?

Answer: 770

Solution: Suppose an ant walks northeast from the point $(-a, a^2)$ on the parabola, where $a \geq 0$. Where will the ant meet the parabola again? It will meet it where the line $y = x + a + a^2$ meets the parabola $y = x^2$ in the first quadrant. We solve the equation $x^2 = x + a + a^2$ to find the x-coordinate, which turns out to be $a + 1$. Thus, the ant meets the parabola again at the point $(a + 1, (a + 1)^2)$. By symmetry, an ant that walks from the point (a, a^2) in the northwesterly direction meets the parabola again at the point $(-(a + 1), (a + 1)^2)$. Also, by symmetry, an ant walking northwest from the point (a, a^2) will meet an ant walking northeast from the point $(-a, a^2)$ at the y-intercept of the line $y = x + a + a^2$, which is $(0, a + a^2)$.

Putting all this together, we see that the ants will meet at the points $(0, n + n^2)$ for each integer $n \geq 0$. Therefore, the diagonals of the squares enclosed by the ant trails have length $(n+1)+(n+1)^2-(n+n^2) = 2(n+1)$. The area of a square with diagonal length $2(n + 1)$ is $2(n + 1)^2$, so the total area of the first 10 squares is

$$\sum_{n=0}^{9} 2(n+1)^2 = 2\frac{(10)(10+1)(2(10)+1)}{6} = \boxed{770},$$

where we have applied the well-known formula for the sum of the first n perfect squares, $\sum_{k=1}^{n} k^2 = \frac{k(k+1)(2k+1)}{6}$.

Problem 6 For each integer from 1 through 2019, Tala calculated the product of its digits. Compute the sum of all 2019 of Tala's products.

Answer: 184,320

Solution: We group together numbers with the same number of digits to compute this sum.

The single-digit numbers 1 through 9 contribute 45 to the sum.

The double-digit numbers 10 through 99 contribute $\sum_{t=1}^{9}\sum_{u=0}^{9} tu = (\sum_{t=1}^{9} t)(\sum_{u=0}^{9} u) = 45^2 = 2{,}025$ to the sum.

The triple-digit numbers 100 through 999 contribute $\sum_{h=1}^{9}\sum_{t=0}^{9}\sum_{u=0}^{9} htu = (\sum_{h=1}^{9} h)(\sum_{t=0}^{9} t)(\sum_{u=0}^{9} u) = 45^3 = 91{,}125$ to the sum.

For the four-digit numbers, note that the numbers from 2000 to 2019 all have a zero digit and do not contribute to the sum. However, since the numbers from 1000 to 1999 all have a 1 in the thousands place, these contribute an amount equivalent to the contribution from the three-digit numbers.

Thus, the answer is $45 + 2025 + 2(91{,}125) = \boxed{184{,}320}$.

Problem 7 Mr. Jones teaches algebra. He has a whiteboard with a pre-drawn coordinate grid that runs from -10 to 10 in both the x and y coordinates. Consequently, when he illustrates the graph of a quadratic, he likes to use a quadratic with the following properties:

I The quadratic sends integers to integers.

II The quadratic has distinct integer roots both between -10 and 10, inclusive.

III The vertex of the quadratic has integer x and y coordinates both between -10 and 10, inclusive.

How many quadratics are there with these properties?

Answer: 478

Solution: A quadratic with distinct roots $a < b$ can be uniquely written as $c(x-a)(x-b)$, where $c \neq 0$. It has vertex $((a+b)/2, -c(b-a)^2/4)$. Condition II forces a and b to be integers between -10 and 10, inclusive. Condition III requires $(a+b)/2$ and $-c(b-a)^2/4)$ to be integers between -10 and 10, inclusive. Thus $a \equiv b \pmod 2$ and $|c(b-a)^2/4| = k$ where k is a positive integer less than 11. In particular, $0 < |c| \leq 40/(b-a)^2$.

Condition I means[2] that there are integers A, B, and C such that

$$c(x-a)(x-b) = Ax(x-1)/2 + Bx + C.$$

[2] George Pólya showed that polynomials that send integers to integers are integral linear combinations of binomial coefficient polynomials. See Pólya, G. (1915), "Über ganzwertige ganze Funktionen," **40**: 1-16.

Comparing coefficients, we have $2c = A$, $-c(a+b) = -A/2+B$, and $cab = C$. Thus, c is either an integer or a half-integer and $B = c(1-a-b)$. Since $a \equiv b \pmod 2$, $1-a-b$ is odd, and since B must be an integer, we conclude that c must also be integer.

Since $40/(b-a)^2$ must be at least 1, there are only 3 possibilities for $b-a$, namely, 2, 4, and 6. When $b-a = 2$, $0 < |c| \leq 10$, which yields 20 possibilities for c. When $b-a = 4$, $0 < |c| \leq 40/16$, which yields 4 possibilities for c. When $b-a = 6$, $0 < |c| \leq 40/36$, which yields 2 possibilities for c. Since there are 19 ways for $b-a = 2$, 17 ways for $b-a = 4$, and 15 ways for $b-a = 6$, the answer is $19(20) + 17(4) + 15(2) = 380 + 68 + 30 = \boxed{478}$.

Problem 8 How many positive integers less than 4000 are not divisible by 2, not divisible by 3, not divisible by 5, and not divisible by 7?

Answer: 913

Solution: The problem is equivalent to asking how many of the positive integers less than 4000 are relatively prime to $2 \times 3 \times 5 \times 7 = 210$. We compute $\phi(210) = 1 \times 2 \times 4 \times 6 = 48$, where ϕ is the Euler ϕ-function. Now $3999 = 210(19) + 9$. Thus, the answer is $19(48)$ plus the number of integers from 1 to 9 that are relatively prime to 210. Of these first 9 integers, only the number 1 is relatively prime to 210. Therefore, the answer is $19(48) + 1 = \boxed{913}$.

Problem 9 Find the least real number K such that for all real numbers x and y, we have $(1+20x^2)(1+19y^2) \geq Kxy$. Express your answer in simplified radical form.

Answer: $-8\sqrt{95}$

Solution: By the AM-GM inequality, we have $1 + 20x^2 \geq 2\sqrt{20}|x|$, with equality iff $1 = 20x^2$. Similarly, $1+19y^2 \geq 2\sqrt{19}|y|$, with equality iff $1 = 19y^2$. Therefore,

$$(1 + 20x^2)(1 + 19y^2) \geq 8\sqrt{95}|xy|, \qquad (*)$$

with equality iff $1 = 20x^2 = 19y^2$. If x and y have opposite sign, this inequality becomes $(1 + 20x^2)(1 + 19y^2) \geq -8\sqrt{95}xy$. If x and y have the same sign, we can drop the absolute values from $*$, and since in this case $8\sqrt{95}xy > -8\sqrt{95}xy$, it is always true that $(1+20x^2)(1+19y^2) \geq -8\sqrt{95}xy$, with equality iff $1 = 20x^2 = 19y^2$ and x and y have opposite sign. We conclude that the answer is $\boxed{-8\sqrt{95}}$.

Problem 10 A 1×5 rectangle is split into five unit squares (cells) numbered 1 through 5 from left to right. A frog starts at cell 1. Every second it jumps from its current cell to one of the adjacent cells. The frog makes exactly 14 jumps. How many paths can the frog take to finish at cell 5?

Answer: 364

Solution: For this problem, it is fairly straightforward to recursively count the number of paths to the various squares after k jumps for $0 \le k \le 14$:

Jumps	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Square 1	1	0	1	0	2	0	5	0	14	0	41	0	122	0	365
Square 2	0	1	0	2	0	5	0	14	0	41	0	122	0	365	0
Square 3	0	0	1	0	3	0	9	0	27	0	81	0	243	0	729
Square 4	0	0	0	1	0	4	0	13	0	40	0	121	0	364	0
Square 5	0	0	0	0	1	0	4	0	13	0	40	0	121	0	364

Thus, the answer is $\boxed{364}$.

Problem 11 Twelve 1's and ten -1's are written on a chalkboard. You select 10 of the numbers and compute their product, then add up these products for every way of choosing 10 numbers from the 22 that are written on the chalkboard. What sum do you get?

Answer: -42

Solution: By Vieta's formulas, the answer is the coefficient of x^{10} in the polynomial $(x+1)^{10}(x-1)^{12} = (x-1)^2(x^2-1)^{10}$. Hence, the answer is $\binom{10}{4} - \binom{10}{5} = 210 - 252 = \boxed{-42}$.

Problem 12 Say that a positive integer is MPR (Math Prize Resolvable) if it can be represented as the sum of a 4-digit number MATH and a 5-digit number PRIZE. (Different letters correspond to different digits. The leading digits M and P can't be zero.) Say that a positive integer is MPRUUD (Math Prize Resolvable with Unique Units Digits) if it is MPR and the set of units digits $\{\mathrm{H}, \mathrm{E}\}$ in the definition of MPR can be uniquely identified. Find the smallest positive integer that is MPR but not MPRUUD.

Answer: 12,843

Solution: Let n be the smallest positive integer that is MPR but not MPRUUD. Let us hope that n has the smallest possible ten-thousands digit, i.e., that its decimal expansion is 1????. Then P = 1. The digit in the thousands place must be at least 2 because M cannot be 0, and it can no longer be 1. Let's hope that n has the smallest possible thousands digits, i.e., that its decimal expansion is 12???. Then M must be 2 and R must be 0. The potentially smallest possible value of n would then be achieved by assigning the two smallest unused digits to A and I. So let us hope that we can set I = 3 and A = 4. (Note that which is assigned the digit 3 and which 4 is immaterial.) Because every pair of unused digits sum to something greater than 9, there will be a carry from the tens to the hundreds place.

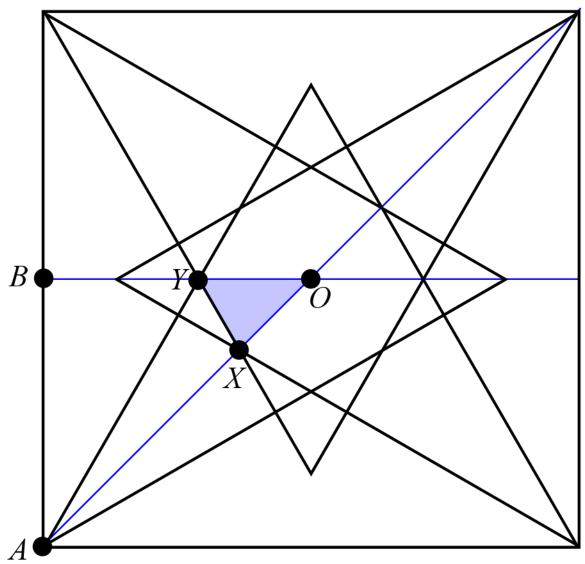
To summarize what we have so far, we hope that n has decimal expansion 128??, while PRIZE is 103?? and MATH is 24??, and the unused digits are 5, 6, 7, 8, and 9.

In order for n to not be MPRUUD, we must be able to get n as a "MATH PRIZE" sum in two different ways that differ in their sets of units digits. Using the unused digits, there are only three ways that different pairs of digits add up to something with the same units digit, namely $5 + 8 = 6 + 7$, $5 + 9 = 6 + 8$, and $6 + 9 = 7 + 8$. Imagine writing down n as a "MATH PRIZE" sum in two different ways, where the two ways have different sets of units digits. Notice that the sum of the tens digits in each way must also add up to the same number. The easiest way to achieve that would be for the sets of units digits and tens digits in the two different ways to swap places, that is, the units digits in one way become the tens digits in the other way. In fact, this must be done, because for each of the three ways $5 + 8 = 6 + 7$, $5 + 9 = 6 + 8$, and $6 + 9 = 7 + 8$, any choice for the units digits does not leave an analogous choice for the tens digits except by taking the other pair of digits that have the same sum as the chosen units digits, as can be seen by inspection. The choice that yields the smallest "MATH PRIZE" sum is the one which involves the smallest common sum, which is $5 + 8 = 6 + 7$. Thus, $n = 10356 + 2487 = 10365 + 2478 = \boxed{12{,}843}$, and all our hopes are justified.

Problem 13 Each side of a unit square (side length 1) is also one side of an equilateral triangle that lies in the square. Compute the area of the intersection of (the interiors of) all four triangles. Express your answer in simplified radical form.

Answer: $\dfrac{9 - 5\sqrt{3}}{3}$ or $3 - \dfrac{5\sqrt{3}}{3}$ or $3 - \dfrac{5}{3}\sqrt{3}$

Solution:



The intersection of all four triangles is an octagon. In the figure, O is the center of the square and B is the midpoint of a side. By symmetry, the area of the octagon is 8 times the area of ΔOXY. Note that $m\angle XOY = 45°$ and $m\angle YAB = 30°$. Also $OY = OB - BY = 1/2 - (\tan 30°)/2$. By the law of sines, $OX/\sin 60° = OY/\sin 75°$. The area of ΔOXY is $\frac{1}{2}OX \cdot OY \sin 45°$. Hence, the area of the octagon is

$$\begin{aligned} 2\sqrt{2}(OX)(OY) &= 2\sqrt{2}\frac{\sin 60°}{\sin 75°}OY^2 \\ &= 2\sqrt{2}\frac{\sin 60°}{\sin 75°}\frac{1}{4}(1 - \tan 30°)^2 \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}/2}{(\sqrt{6}+\sqrt{2})/4}(1 - 1/\sqrt{3})^2\,, \end{aligned}$$

which simplifies to $\boxed{\dfrac{9 - 5\sqrt{3}}{3}}$.

Problem 14 Devah draws a row of 1000 equally spaced dots on a sheet of paper. She goes through the dots from left to right, one by one, checking if the midpoint between the current dot and some remaining dot to its left is also a remaining dot. If so, she erases the current dot. How many dots does Devah end up erasing?

Answer: 895

Solution: Label the dots 0 through 999, from left to right. We claim that the dots that are erased are exactly those labeled by a number whose ternary expansion has a 2-digit. To see this, we proceed by induction. The dot labeled 0 remains, vacuously. Now suppose Devah has made it through dot $k-1$ and has so far erased exactly those labeled by a number whose ternary expansion has a 2-digit. To determine if dot k should be erased or not, we must check to see if there exists unerased dots $A < B$ such that $2B = A + k$.

Suppose the ternary expansion of k has a 2-digit. Let A be the number whose ternary expansion has 1-digits in the same places where the ternary expansion of k also has a 1-digit, and place 0-digits in all other places. Then $A < k$ and $k + A$ has a ternary expansion with no 1-digit. Thus, $(k + A)/2$ is a whole number with a ternary expansion with no 2-digit, so corresponds to an unerased dot at the midpoint of A and k.

Suppose the ternary expansion of k does not have a 2-digit and that there exist unerased dots A and B with $A < B$ and $2B = A + k$. By induction, the ternary expansion of B has no digit of 2, hence, the ternary expansion of $2B = A + k$ has no 1-digit. However, note that when adding A and k in ternary, there can be no carries, since both only have 0 and 1-digits in their ternary expansions. The only way for $A + k$ to have no 1-digits in ternary would be for every digit of 1 in the ternary expansion of A to match with a digit of 1 in the same place in the ternary expansion of k, and vice versa. But this means $A = k$, which is not possible since $A < B = (A+k)/2$, which implies that $A < k$.

Since $999 = 1101000_3$, there are $1 + 1101000_2 = 105$ dots that remain and $1000 - 105 = \boxed{895}$ dots that were erased.

Problem 15 How many ordered pairs (x, y) of real numbers x and y are there such that $-100\pi \le x \le 100\pi$, $-100\pi \le y \le 100\pi$, $x + y = 20.19$, and $\tan x + \tan y = 20.19$?

Answer: 388

Solution: Suppose that $x + y = 20.19$ and $\tan x + \tan y = 20.19$. By the tangent addition formula,

$$\tan 20.19 = \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{20.19}{1 - \tan x \tan y}.$$

Rearranging gives $\tan x \tan y = 1 - \frac{20.19}{\tan 20.19}$. So we know the sum and product of $\tan x$ and $\tan y$. In particular, $\tan x$ and $\tan y$ are the roots of the quadratic $t^2 - 20.19t + 1 - \frac{20.19}{\tan 20.19}$. We will show later that $\tan 20.19$ is strictly between 0 and 20.19. That means the constant coefficient of the quadratic is negative. Hence the quadratic has one negative root (call it r) and one positive root (call it s).

Our problem is to count the pairs (x, y) with x and y in the interval $[-100\pi, 100\pi]$ such that $x + y = 20.19$ and $(\tan x, \tan y)$ equal to either (r, s) or (s, r). By symmetry, we will first count solutions to $(\tan x, \tan y) = (r, s)$ and then double the count at the end.

Let's eliminate y using the equation $y = 20.19 - x$. The two conditions that x and y are in the interval $[-100\pi, 100\pi]$ become the single condition that x is in the interval $[20.19 - 100\pi, 100\pi]$. If $x + y = 20.19$ and $\tan x = r$, then $\tan y = s$, by tangent addition. So we merely have to count the number of x in the interval $[20.19 - 100\pi, 100\pi]$ such that $\tan x = r$.

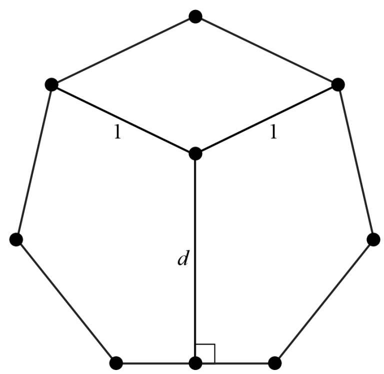
Because $3.14 < \pi < 3.2$, we have $6\pi < 20.19 < 6.5\pi$. Hence $20.19 - 100\pi$ is in the interval $(-94\pi, -93.5\pi)$. Because tan is positive on that interval, our problem is equivalent to counting x in the interval $(-94\pi, 100\pi]$ such that $\tan x = r$. We can split $(-94\pi, 100\pi]$ into 194 subintervals of length π, each of which has exactly one solution to $\tan x = r$. So the number of solutions is 194.

Remember that we have to double the number of solutions. So the total count is $2 \cdot 194$, or $\boxed{388}$.

The only thing remaining is to check that $\tan 20.19$ is strictly between 0 and 20.19. Recall that $6\pi < 20.19 < 6.5\pi$. That means $\tan 20.19$ is positive. Because $\pi > 3.14$, we have $6.5\pi > 20.41 > 20.19 + 0.2$. In particular, $0.2 < 6.5\pi - 20.19 < 0.5\pi$. Because $\tan z \geq z$ for $0 \leq z < 0.5\pi$, we have $\tan(6.5\pi - 20.19) \geq 0.2$. Hence we have

$$\tan 20.19 = \frac{1}{\tan(0.5\pi - 20.19)} = \frac{1}{\tan(6.5\pi - 20.19)} \leq 5 < 20.19.$$

Problem 16 The figure shows a regular heptagon with sides of length 1.



Determine the indicated length d. Express your answer in simplified radical form.

Answer: $\dfrac{\sqrt{7}}{2}$

Solution: Place the figure in the complex plane with the two lower vertices at -1 and 0. Let $w = e^{2\pi i/7} + e^{4\pi i/7} + e^{8\pi i/7}$. Note that $w = -1/2 + di$.

Let $f(x) = (x - e^{2\pi i/7})(x - e^{4\pi i/7})(x - e^{8\pi i/7})$. Observe that $(x-1)f(x)\overline{f(x)} = x^7 - 1$. Divide this equation by $x - 1$ on both sides to obtain

$$f(x)\overline{f(x)} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6.$$

Substitute 1 for x in this last equation to find $|f(1)|^2 = 7$. The coefficient of x in $f(x)$ is $e^{12\pi i/7} + e^{10\pi i/7} + e^{6\pi i/7} = e^{-2\pi i/7} + e^{-4\pi i/7} + e^{-8\pi i/7} = \overline{w} = -1/2 - di$. Thus, $f(x) = x^3 - (-1/2 + di)x^2 + (-1/2 - di)x - 1$, so that $|f(1)| = 2d$. Hence, $d = \boxed{\dfrac{\sqrt{7}}{2}}$.

Problem 17 Let P be a right prism whose two bases are equilateral triangles with side length 2. The height of P is $2\sqrt{3}$. Let l be the line connecting the centroids of the bases. Remove the solid, keeping only the bases. Rotate one of the bases 180° about l. Let T be the convex hull of the two current triangles. What is the volume of T?

Answer: 8

Solution: Note that the convex hull of the two triangles is the convex hull of their 6 vertices. Observe that the 6 vertices are the vertices of an octahedron, although not a regular one. But we can make it regular by scaling along l.

Consider the regular octahedron whose vertices are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$. In this regular octahedron, the side length is $\sqrt{2}$ and the distance between parallel equilateral triangular faces is the distance between the planes $x+y+z=1$ and $x+y+z=-1$, which is $2/\sqrt{3}$. Also note that the line parameterized by (t, t, t) is perpendicular to these planes and connects the centroids ($(1/3, 1/3, 1/3)$ and $(-1/3, -1/3, -1/3)$) of the faces the line passes through. Thus, we can make the given object a regular octahedron by scaling along l so that it has length $2\sqrt{2/3}$, i.e., by a scale factor of $\sqrt{2}/3$.

The volume of a regular octahedron with side length s is $\sqrt{2}s^3/3$, so the volume of our scaled object is $8\sqrt{2}/3$. To recover the volume of the original object, we divide by $\sqrt{2}/3$ to find that the answer is $\boxed{8}$.

Problem 18 How many ordered triples (a, b, c) of integers with $-15 \leq a, b, c \leq 15$ are there such that the three equations $ax + by = c$, $bx + cy = a$, and $cx + ay = b$ correspond to lines that are distinct and concurrent?

Answer: 720

Solution: Replace 15 in the problem statement with n.

We partition the desired set of lattice points according to the value of a.

If $a = 0$, then the lines are $by = c$, $bx + cy = 0$, and $cx = b$. So b and c cannot be zero. Concurrency implies that $b^2/c + c^2/b = 0$, i.e., $b = -c$. This gives the $2n$ solutions $(0, k, -k)$, where $k = -n, \ldots, n$, with $k \neq 0$.

If $a = 1$, the lines are $x + by = c$, $bx + cy = 1$, and $cx + y = b$. These lines are distinct and nonparallel provided that $c \neq b^2$, $b \neq c^2$, and $1 \neq bc$. If these conditions hold, then the intersection of the first and third lines is $((c - b^2)/(1 - bc), (b - c^2)/(1 - bc))$, and this is a solution of the second line if and only if $b(c - b^2) + c(b - c^2) = 1 - bc$. This is equivalent to $(b + c + 1)(b^2 + c^2 - bc - b - c + 1) = 0$. The second factor is equal to $(b + c - 2)^2 - 3(b - c)^2$, which cannot be 0 unless $b = c = 1$, since $\sqrt{3}$ is irrational. However, if $b = c = 1$, then the lines are parallel. Therefore, the only solutions in the case $a = 1$ satisfy $b + c + 1 = 0$. This accounts for the $2n$ solutions $(1, k, -k - 1)$, where $k = -n, \ldots, n - 1$.

When $a > 1$, dividing all coefficients by a doesn't change their geometric nature, hence we have the solutions $(a, ak, -a(k + 1))$, where k ranges from $-n/a$ to $(n - a)/a$ in steps of $1/a$. This yields $2n - a + 1$ solutions.

When $a < 0$, we can multiply throughout by -1.

Thus, there are $2n + 2(\sum_{k=0}^{n-1}(2n - k)) = 2n + n(3n + 1) = 3n(n + 1)$. Substitute 15 for n to find that the answer is $\boxed{720}$.

Problem 19 Consider the base 27 number

$$n = ABCDEFGHIJKLMNOPQRSTUVWXYZ,$$

where each letter has the value of its position in the alphabet. What remainder do you get when you divide n by 100? (The remainder is an integer between 0 and 99, inclusive.)

Answer: 25

Solution: We apply the Chinese remainder theorem to find the answer by computing n modulo 4 and 25.

By going through the digits of n from right to left, we find that

$$n \equiv 26 - 25 + 24 - 23 + 22 - 21 + \cdots + 4 - 3 + 2 - 1 \equiv 13 \equiv 1 \pmod 4.$$

Note that $n + 1 = ABCDEFGHIJKLMNOPQRSTUVWXZ0_{27}$. By direct computation, we find

$$26(n + 1) = AAAAAAAAAAAAAAAAAAAAAAAAAA0_{27},$$

where the base 27 number has 26 digits of A. Hence, $26(n+1)+1 = \frac{27^{27}-1}{26}$. Reducing these equation modulo 25, we find $n + 2 \equiv 2^{27} - 1 \pmod{25}$. By Euler's generalization of Fermat's little theorem, $2^{27} \equiv 2^7 \equiv 3 \pmod{25}$. Thus we find $n \equiv 0 \pmod{25}$.

Hence, $n \equiv 25 \pmod{100}$, and the answer is $\boxed{25}$.

Problem 20 Evaluate the infinite product

$$\prod_{k=2}^{\infty} \left(1 - 4\sin^2 \frac{\pi}{3 \cdot 2^k}\right).$$

Express your answer as a fraction in simplest form.

Answer: $\dfrac{2}{3}$

Solution: In this solution, we will make use of the triple angle formulas

$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$$

and

$$\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta).$$

Observe that

$$\begin{aligned} 1 - 4\sin^2(\frac{\pi}{3 \cdot 2^k}) &= 4\cos^2(\frac{\pi}{3 \cdot 2^k}) - 3 \\ &= \frac{4\cos^3(\frac{\pi}{3 \cdot 2^k}) - 3\cos(\frac{\pi}{3 \cdot 2^k})}{\cos(\frac{\pi}{3 \cdot 2^k})} \\ &= \frac{\cos(\frac{\pi}{2^k})}{\cos(\frac{\pi}{3 \cdot 2^k})}. \end{aligned}$$

Since $\sin(2x) = 2\sin(x)\cos(x)$, we have $\cos x = \frac{\sin(2x)}{2\sin x}$. Therefore,

$$\prod_{k=2}^{n} \cos(\frac{\theta}{2^k}) = \prod_{k=2}^{n} \frac{\sin(\frac{\theta}{2^{k-1}})}{2\sin(\frac{\theta}{2^k})} = \frac{\sin(\theta/2)}{2^{n-1}\sin(\theta/2^n)}.$$

Hence,

$$\begin{aligned} \prod_{k=2}^{n} \left(1 - 4\sin^2 \frac{\pi}{3 \cdot 2^k}\right) &= \prod_{k=2}^{n} \frac{\cos(\frac{\pi}{2^k})}{\cos(\frac{\pi}{3 \cdot 2^k})} \\ &= \frac{\frac{\sin(\pi/2)}{2^{n-1}\sin(\pi/2^n)}}{\frac{\sin((\pi/3)/2)}{2^{n-1}\sin((\pi/3)/2^n)}} \\ &= 2\frac{\sin((\pi/3)/2^n)}{\sin(\pi/2^n)} \\ &= 2\frac{\sin((\pi/3)/2^n)}{3\sin((\pi/3)/2^n) - 4\sin^3((\pi/3)/2^n)} \\ &= \frac{2}{3 - 4\sin^2((\pi/3)/2^n)}. \end{aligned}$$

As n tends to ∞, this last expression tends to $\boxed{\frac{2}{3}}$, which is the answer.