Advantage Testing Foundation

The Thirteenth Annual Math Prize for Girls

Sunday, October 10, 2021

Test Version A

DIRECTIONS

1. Do not open this test until your proctor instructs you to.
2. Fill out the top of your answer sheet. Your student number is on your name tag. Your test version (A, B, C, or D) is above on this page.
3. You may use pencils, pens, and erasers. You may also use the scratch paper that we provide. You may NOT use calculators, rulers, protractors, compasses, graph paper, or books.
4. Write your final answers on the answer sheet. When a problem specifies a particular form for the answer, write your answer in that form. Write your answers clearly. We will collect only the answer sheet from you.

NOTES

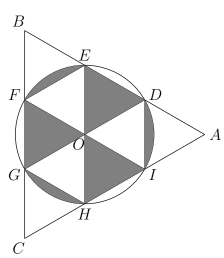
1. This test contains 20 problems. You will have 150 minutes (2.5 hours) to take the test. Your score will be the number of correct answers.
2. Figures are not necessarily drawn to scale.
3. Good luck!

Math Prize for Girls 2021

1. A soccer coach named C does a header drill with two players A and B, but they all forgot to put sunscreen on their foreheads. They solve this issue by dunking the ball into a vat of sunscreen before starting the drill. Coach C heads the ball to A, who heads the ball back to C, who then heads the ball to B, who heads the ball back to C; this pattern $CACBCACB\ldots$ repeats ad infinitum. Each time a person heads the ball, $1/10$ of the sunscreen left on the ball ends up on the person's forehead. In the limit, what fraction of the sunscreen originally on the ball will end up on the coach's forehead? Express your answer as a fraction in simplest form.

2. Let m and n be positive integers such that $m^4 - n^4 = 3439$. What is the value of mn?

3. Let O be the center of an equilateral triangle ABC of area $1/\pi$. As shown in the diagram, a circle centered at O meets the triangle at points D, E, F, G, H, and I, which trisect each of the triangle's sides.



Compute the total area of all six shaded regions. Express your answer in simplified radical form.

4. For a positive integer n, let $v(n)$ denote the largest integer j such that n is divisible by 2^j. Let a and b be chosen uniformly and independently at random from among the integers between 1 and 32, inclusive. What is the probability that $v(a) > v(b)$? Express your answer as a fraction in simplest form.

5. Among all fractions (whose numerator and denominator are positive integers) strictly between $\frac{6}{17}$ and $\frac{9}{25}$, which one has the smallest denominator?

6. The number 734,851,474,594,578,436,096 is equal to n^6 for some positive integer n. What is the value of n?

7. Compute the value of the infinite series

$$\sum_{k=0}^{\infty} \frac{\cos(k\pi/4)}{2^k}.$$

Express your answer in simplified radical form.

8. In $\triangle ABC$, let point D be on $\overline{BC}$ such that the perimeters of $\triangle ADB$ and $\triangle ADC$ are equal. Let point E be on $\overline{AC}$ such that the perimeters of $\triangle BEA$ and $\triangle BEC$ are equal. Let point F be the intersection of $\overline{AB}$ with the line that passes through C and the intersection of $\overline{AD}$ and $\overline{BE}$. Given that $BD = 10$, $CD = 2$, and $BF/FA = 3$, what is the perimeter of $\triangle ABC$?

9. Let H be a regular hexagon with area 360. Three distinct vertices X, Y, and Z are picked randomly, with all possible triples of distinct vertices equally likely. Let A, B, and C be the unpicked vertices. What is the expected value (average value) of the area of the intersection of $\triangle ABC$ and $\triangle XYZ$?

10. Let P be the product of all the entries in row 2021 of Pascal's triangle (the row that begins 1, 2021, . . .). What is the largest integer j such that P is divisible by 101^j?

11. Say that a sequence a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 is *cool* if

- the sequence contains each of the integers 1 through 8 exactly once, and
- every pair of consecutive terms in the sequence are relatively prime. In other words, a_1 and a_2 are relatively prime, a_2 and a_3 are relatively prime, . . . , and a_7 and a_8 are relatively prime.

How many cool sequences are there?

12. Let P_1, P_2, P_3, P_4, P_5, and P_6 be six parabolas in the plane, each congruent to the parabola $y = x^2/16$. The vertices of the six parabolas are evenly spaced around a circle. The parabolas open outward with their axes being extensions of six of the circle's radii. Parabola P_1 is tangent to P_2, which is tangent to P_3, which is tangent to P_4, which is tangent to P_5, which is tangent to P_6, which is tangent to P_1. What is the diameter of the circle?

13. There are 2021 light bulbs in a row, labeled 1 through 2021, each with an on/off switch. They all start in the off position when 1011 people walk by. The first person flips the switch on every bulb; the second person flips the switch on every 3rd bulb (bulbs 3, 6, etc.); the third person flips the switch on every 5th bulb; and so on. In general, the kth person flips the switch on every $(2k-1)$th light bulb, starting with bulb $2k-1$. After all 1011 people have gone by, how many light bulbs are on?

14. Let S be the set of monic polynomials in x of degree 6 all of whose roots are members of the set $\{-1, 0, 1\}$. Let P be the sum of the polynomials in S. What is the coefficient of x^4 in $P(x)$?

15. There are 300 points in space. Four planes A, B, C, and D each have the property that they split the 300 points into two equal sets. (No plane contains one of the 300 points.) What is the maximum number of points that can be found inside the tetrahedron whose faces are on A, B, C, and D?

16. Let G be the set of points (x, y) such that x and y are positive integers less than or equal to 20. Say that a ray in the coordinate plane is *ocular* if it starts at $(0, 0)$ and passes through at least one point in G. Let A be the set of angle measures of acute angles formed by two distinct ocular rays. Determine

$$\min_{a \in A} \tan a.$$

Express your answer as a fraction in simplest form.

17. In the coordinate plane, let $A = (-8, 0)$, $B = (8, 0)$, and $C = (t, 6)$. What is the maximum value of $\sin m\angle CAB \cdot \sin m\angle CBA$, over all real numbers t? Express your answer as a fraction in simplest form.

18. Let N be the set of square-free positive integers less than or equal to 50. (A *square-free* number is an integer that is not divisible by a perfect square bigger than 1.) How many 3-element subsets S of N are there such that the greatest common divisor of all 3 numbers in S is 1, but no pair of numbers in S is relatively prime?

19. Let T be a regular tetrahedron. Let t be the regular tetrahedron whose vertices are the centers of the faces of T. Let O be the circumcenter of either tetrahedron. Given a point P different from O, let $m(P)$ be the midpoint of the points of intersection of the ray $\overrightarrow{OP}$ with t and T. Let S be the set of eight points $m(P)$ where P is a vertex of either t or T. What is the volume of the convex hull of S divided by the volume of t? Express your answer as a fraction in simplest form.

20. Let G be the set of points (x, y) such that x and y are positive integers less than or equal to 6. A *magic grid* is an assignment of an integer to each point in G such that, for every square with horizontal and vertical sides and all four vertices in G, the sum of the integers assigned to the four vertices is the same as the corresponding sum for any other such square. A magic grid is formed so that the product of all 36 integers is the smallest possible value greater than 1. What is this product?