The Advantage Testing Foundation

Math Prize for Girls at MIT

2021 Solutions

Problem 1 A soccer coach named C does a header drill with two players A and B, but they all forgot to put sunscreen on their foreheads. They solve this issue by dunking the ball into a vat of sunscreen before starting the drill. Coach C heads the ball to A, who heads the ball back to C, who then heads the ball to B, who heads the ball back to C; this pattern $CACBCACB\ldots$ repeats ad infinitum. Each time a person heads the ball, $1/10$ of the sunscreen left on the ball ends up on the person's forehead. In the limit, what fraction of the sunscreen originally on the ball will end up on the coach's forehead? Express your answer as a fraction in simplest form.

Answer: $\dfrac{10}{19}$

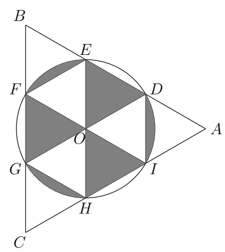
Solution: The problem is equivalent to having 4 people, labeled 1, 2, 3, and 4, heading the ball in the cycle 1, 2, 3, 4, 1, 2, 3, 4, etc., then combining the amount of sunscreen on the first and third player's heads. Let A_k be the fraction of sunscreen on player k's head in the limit. By symmetry, $A_{k+1} = (9/10)A_k$, for $k = 1$, 2, or 3. Also $A_1 + A_2 + A_3 + A_4 = 1$. Thus, $A_1(1 + 9/10 + (9/10)^2 + (9/10)^3) = 1$, from which we find $A_1 = 1000/(1000 + 900 + 810 + 729) = 1000/3439$. Therefore, the answer is $A_1 + (9/10)^2 A_1 = 1000/3439(181/100) = \boxed{10/19}$.

Problem 2 Let m and n be positive integers such that $m^4 - n^4 = 3439$. What is the value of mn?

Answer: 90

Solution: The prime factorization of 3439 is (19)(181). Also, $m^4 - n^4 = (m^2 - n^2)(m^2 + n^2) = (m - n)(m + n)(m^2 + n^2)$. Since $m > n \geq 1$, we must have $m^2 + n^2 > m + n > 1$. Therefore, $m - n = 1$, $m + n = 19$, and $m^2 + n^2 = 181$. Solving the first two equations yields $m = 10$ and $n = 9$, and since $10^2 + 9^2 = 181$, the answer is $10 \times 9 = \boxed{90}$.

Problem 3 Let O be the center of an equilateral triangle ABC of area $1/\pi$. As shown in the diagram, a circle centered at O meets the triangle at points D, E, F, G, H, and I, which trisect each of the triangle's sides.



Compute the total area of all six shaded regions. Express your answer in simplified radical form.

Answer: $\dfrac{2\sqrt{3}}{27}$

Solution: Let s be the side length of the equilateral triangle, so $\frac{\sqrt{3}}{4}s^2 = 1/\pi$. Since $BE : BA = CH : CA = 1/3$, we know that $\overline{EH} \mid\mid \overline{BC}$. Similarly $\overline{GD} \mid\mid \overline{CA}$ and $\overline{JF} \mid\mid \overline{AB}$. Since the centroid of any triangle is $1/3$ of the way up the length of any median, $\overline{EH}$ contains O and is, therefore, a diameter of the circle. Hence, $OE = OH = 1/2(EH) = 1/2(2/3s) = s/3$ and the angles formed by $\overline{EH}$, $\overline{FI}$, and $\overline{GD}$ at O all measure $60°$, dividing the circle into 6 congruent sectors. By rotating regular hexagon $DEFGHI$ by $60°$, we see that the shaded regions amount to half the circle. Thus, the area of the shaded region is

$$\frac{1}{2}\pi(OE)^2 = \frac{1}{2}\pi(s/3)^2 = \frac{1}{2}\pi(\frac{4}{\pi\sqrt{3}})/9 = \boxed{\frac{2\sqrt{3}}{27}}.$$

Problem 4 For a positive integer n, let $v(n)$ denote the largest integer j such that n is divisible by 2^j. Let a and b be chosen uniformly and independently at random from among the integers between 1 and 32, inclusive. What is the probability that $v(a) > v(b)$? Express your answer as a fraction in simplest form.

Answer: $\dfrac{341}{1024}$

Solution: If p is the probability that $v(a) = v(b)$, then the answer is $(1-p)/2$, by symmetry. We compute p more generally for integers chosen from between

1 and 2^n, inclusive. Note that for a fixed $0 \leq k < n$, the number of integers x between 1 and 2^n, inclusive, such that $v(x) = k$ is equal to 2^{n-k-1}, and that there is only 1 such integer (namely 2^n) such that $v(x) = n$. We compute

$$\begin{aligned} p &= \frac{1}{2^{2n}} \sum_{k=0}^{n} \#\{x \mid v(x) = k\}^2 \\ &= \frac{1}{2^{2n}}(1 + \sum_{k=0}^{n-1} 2^{2(n-k-1)}) \\ &= \frac{1}{2^{2n}}(1 + \frac{4^n - 1}{4 - 1}), \end{aligned}$$

from which we find the desired probability $(1 - p)/2 = \frac{4^n-1}{3(4^n)}$. Substituting 5 for n, we find that the answer is $\boxed{341/1024}$.

Problem 5 Among all fractions (whose numerator and denominator are positive integers) strictly between $\frac{6}{17}$ and $\frac{9}{25}$, which one has the smallest denominator?

Answer: $\dfrac{5}{14}$

Solution: The mediant of $6/17$ and $9/25$ is $(6+9)/(17+25) = 15/42 = 5/14$. The mediant of two fractions always lies between the two, or one can check directly that $6/17 < 5/14 < 9/25$.

We claim that $5/14$ is the fraction with smallest denominator between $6/17$ and $9/25$. To see this, suppose that p and q are whole numbers with $q < 14$ and $p/q \neq 5/14$. Then

$$\left|\frac{p}{q} - \frac{5}{14}\right| = \frac{|14p - 5q|}{14q} \geq \frac{1}{14q} \geq \frac{1}{14 \cdot 13}.$$

However,

$$\frac{5}{14} - \frac{6}{17} = \frac{1}{14 \cdot 17} < \frac{1}{14 \cdot 13},$$

and

$$\frac{9}{25} - \frac{5}{14} = \frac{1}{14 \cdot 25} < \frac{1}{14 \cdot 13}.$$

Therefore $p/q \notin [6/17, 9/25]$, and the answer is $\boxed{\dfrac{5}{14}}$.

Problem 6 The number 734,851,474,594,578,436,096 is equal to n^6 for some positive integer n. What is the value of n?

Answer: 3004

Solution: Let x be the given number. Note that x is slightly bigger than $3000^6 = 729 \times 10^{18}$. Focusing on higher order digits, by the binomial theorem, we know that

$$(3000 + a)^6 > 3000^6 + 6(3000^5)a = 729(10^{18}) + 1458a(10^{15}).$$

Since $x - 729(10^{18}) < 6(10^{18})$, we know that $a < 5$. Focusing on lower order digits, again, by the binomial theorem, we know that

$$(3000 + a)^6 = 1000y + a^6,$$

for some integer y. Therefore, the units, tens, and hundreds digits of $(3000 + a)^6$ are the same as those of a^6. Of the numbers 1, 2, 3, and 4, only 4^6 ends with the digits 096. Therefore, the answer is $\boxed{3004}$.

Problem 7 Compute the value of the infinite series

$$\sum_{k=0}^{\infty} \frac{\cos(k\pi/4)}{2^k}.$$

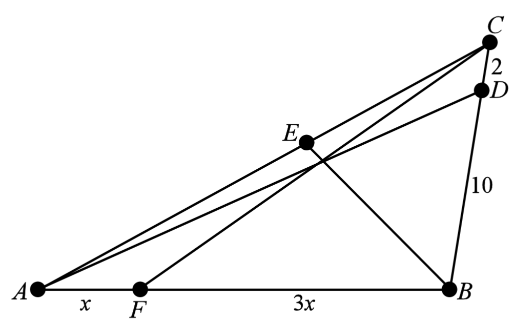
Express your answer in simplified radical form.

Answer: $\dfrac{3\sqrt{2} + 16}{17}$

Solution: The function $\cos(\pi x/4)$ has period 8. By grouping together every 8 terms, we recognize that the sum is a geometric series with common ratio

2^{-8}. Therefore, the answer is

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{\cos(k\pi/4)}{2^k} &= (\sum_{k=0}^{7} \cos(k\pi/4)/2^k)(1 + 2^{-8} + 2^{-16} + 2^{-32} + \dots) \\
&= \frac{\sum_{k=0}^{7} \cos(k\pi/4)/2^k}{1 - 2^{-8}} \\
&= \frac{1/1 + \sqrt{2}/4 - \sqrt{2}/16 - 1/16 - \sqrt{2}/64 + \sqrt{2}/256}{1 - 2^{-8}} \\
&= \frac{15/16 + 45\sqrt{2}/256}{1 - 2^{-8}} \\
&= \boxed{\frac{3\sqrt{2} + 16}{17}}.
\end{aligned}$$



(not to scale)

Problem 8 In $\triangle ABC$, let point D be on $\overline{BC}$ such that the perimeters of $\triangle ADB$ and $\triangle ADC$ are equal. Let point E be on $\overline{AC}$ such that the perimeters of $\triangle BEA$ and $\triangle BEC$ are equal. Let point F be the intersection of $\overline{AB}$ with the line that passes through C and the intersection of $\overline{AD}$ and $\overline{BE}$. Given that $BD = 10$, $CD = 2$, and $BF/FA = 3$, what is the perimeter of $\triangle ABC$?

Answer: 36

Solution: See the figure. Because $\overline{AD}$ and $\overline{BE}$ both bisect the perimeter of $\triangle ABC$, we must have $AE = BD = 10$. By Ceva's theorem,

$$\frac{BF}{FA}\frac{AE}{EC}\frac{CD}{DB} = 1.$$

Substituting the known information into this equation yields

$$3 \cdot \frac{10}{EC}\frac{2}{10} = 1,$$

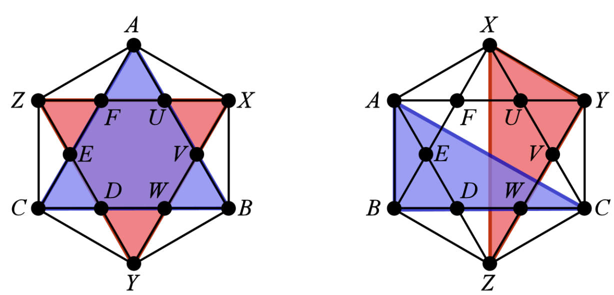
from which we find $EC = 6$. Since $\overline{AD}$ is a perimeter bisector, $AE + EC + CD = 10 + 6 + 2 = 18$ must be half the perimeter of $\triangle ABC$. Hence, the perimeter of $\triangle ABC$ is $2(18)$ or $\boxed{36}$.

Note: Although we did not explicitly use it, it is true that the Cevians that bisect the perimeter of a triangle are concurrent.

Problem 9 Let H be a regular hexagon with area 360. Three distinct vertices X, Y, and Z are picked randomly, with all possible triples of distinct vertices equally likely. Let A, B, and C be the unpicked vertices. What is the expected value (average value) of the area of the intersection of $\triangle ABC$ and $\triangle XYZ$?

Answer: 24

Solution: Let s be the side length of H. There are 20 ways to pick 3 of the 6 vertices of a hexagon. Of these, there are 3 possibilities: (1) there are no pairs of adjacent vertices, which accounts for 2 of the ways, (2) there is a single pair of adjacent vertices, which accounts for 12 of the ways, and (3) the 3 vertices are consecutive, which accounts for 6 of the ways.



In the figure above, H is the larger hexagon in each picture.

In Case 1, illustrated above left, the two triangles $\triangle XYZ$ and $\triangle ABC$ intersect in regular hexagon $UVWDEF$. Note that the distance between $\overline{UF}$ and $\overline{DW}$ is equal to s. The corresponding length in H is the distance between $\overline{AX}$ and $\overline{YC}$, which is $\sqrt{3}s$. Therefore, the area of hexagon $UVWDEF$ is $(\frac{s}{\sqrt{3}s})^2 360 = 120$.

In Case 2, illustrated above right, the two triangles $\triangle XYZ$ and $\triangle ABC$ instersect in a region which is obtained by slicing out a piece of hexagon $UVWDEF$ by perpendicular bisectors to adjacent sides and so the intersection has area one-sixth that of hexagon $UVWDEF$, or 20.

In Case 3, triangles $\triangle XYZ$ and $\triangle ABC$ do not intersect.

The expected area is therefore $(\frac{2}{20})120 + (\frac{12}{20})20 = \boxed{24}$.

Problem 10 Let P be the product of all the entries in row 2021 of Pascal's triangle (the row that begins 1, 2021, ...). What is the largest integer j such that P is divisible by 101^j?

Answer: 1980

Solution: The kth entry in the nth row of Pascal's triangle is $\frac{n!}{k!(n-k)!}$. If we simply multiply these fractions without cancellation, we obtain a fraction with a numerator and denominator that are a product of numbers between 1 and n, inclusive. The number of times k appears in the numerator is $n+1$. The number of times k appears in the denominator is $2(n-k+1)$. Therefore,

$$P = \prod_{k=1}^{n} k^{(n+1)-2(n-k+1)} = \prod_{k=1}^{n} k^{2k-n-1}.$$

Since 101 is a prime number and $2021 < 101^2$, we find that

$$j = \sum_{i=1}^{\lfloor 2021/101 \rfloor} (2(101i) - 2021 - 1).$$

Thus, $j = \sum_{i=1}^{20}(202i - 2022) = 202\frac{20(21)}{2} - 2022(20) = \boxed{1980}$.

Problem 11 Say that a sequence a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 is *cool* if

- the sequence contains each of the integers 1 through 8 exactly once, and

- every pair of consecutive terms in the sequence are relatively prime. In other words, a_1 and a_2 are relatively prime, a_2 and a_3 are relatively prime, ..., and a_7 and a_8 are relatively prime.

How many cool sequences are there?

Answer: 1728

Solution: By considering separately the 4 prime numbers that appear as factors of the numbers 1 through 8, we find that a sequence is cool if and only if even numbers are separated by at least one odd number and the 3 and 6 are not adjacent.

Since a cool sequence has 3 pairs of consecutive (in terms of the sequence, not numerical order) even numbers, we require at least 3 odd numbers to keep consecutive pairs separated. This leaves one odd number which may be placed between any two consecutive even numbers, or before the first, or after the last. Therefore, the evens and odds in a cool sequence fall into one of the following 5 patterns:

$$
\begin{gathered}
OEOEOEOE \\
EOOEOEOE \\
EOEOOEOE \\
EOEOEOOE \\
EOEOEOEO
\end{gathered}
$$

As we fill in each pattern with specific numbers, we have to pay special attention to where we place the 6, because the location of 6 limits the location of the 3. If 6 is the second or third even number, it must have an odd number both before and after, leaving only 2 locations for the 3. If t is the first or fourth even number, it will be adjacent to only one odd number, unless it is the first even number in the first pattern above or the fourth even number in the last pattern above, in which case it will be preceded and succeeded by an odd number. This gives us a total of $9+10+10+10+9=48$ patterns in which the positions of 3 and 6 are set, and the locations of the remaining odd and even numbers are determined only up to parity. For each such pattern, there are $3! \cdot 3! = 36$ ways to place the remaining numbers. We conclude that there are $36 \cdot 48 = \boxed{1728}$ cool sequences.

Problem 12 Let P_1, P_2, P_3, P_4, P_5, and P_6 be six parabolas in the plane, each congruent to the parabola $y = x^2/16$. The vertices of the six parabolas are evenly spaced around a circle. The parabolas open outward with their axes being extensions of six of the circle's radii. Parabola P_1 is tangent to P_2, which is tangent to P_3, which is tangent to P_4, which is tangent to P_5, which is tangent to P_6, which is tangent to P_1. What is the diameter of the circle?

Answer: 24

Solution: If the center of the circle is placed at the origin of coordinates, then one of the parabolas can be placed vertically so that it rests upon the lines $y = \pm(\tan 60^\circ)x = \pm\sqrt{3}x$. The equation of this parabola has the form $y = x^2/16 + a$. To find a, we insist that the simultaneous equations $y = \sqrt{3}x$ and $y = x^2/16 + a$ have a single solution, that is, the quadratic $x^2/16 - \sqrt{3}x + a = 0$ must have a double root. Therefore, its discriminant must be zero, i.e., $3 - a/4 = 0$, or $a = 12$. Since the vertex of $y = x^2/16 + 12$ is $(0, 12)$, we conclude that the diameter of the circle is $2(12) = \boxed{24}$.

Problem 13 There are 2021 light bulbs in a row, labeled 1 through 2021, each with an on/off switch. They all start in the off position when 1011 people walk by. The first person flips the switch on every bulb; the second person flips the switch on every 3rd bulb (bulbs 3, 6, etc.); the third person flips the switch on every 5th bulb; and so on. In general, the kth person flips the switch on every $(2k-1)$th light bulb, starting with bulb $2k-1$. After all 1011 people have gone by, how many light bulbs are on?

Answer: 75

Solution: If light bulb m is left on, it means that m has an odd number of odd factors. In other words, m is either a perfect square or twice a perfect square. No number is both a perfect square and twice a perfect square, so we can count such m by counting perfect squares and twice perfect squares separately, then adding. The number of perfect squares less than or equal to 2021 is $\lfloor\sqrt{2021}\rfloor = 44$. The number of twice perfect squares less than or equal to 2021 is $\lfloor\sqrt{2021/2}\rfloor = 31$. Therefore, the answer is $44 + 31 = \boxed{75}$.

Problem 14 Let S be the set of monic polynomials in x of degree 6 all of whose roots are members of the set $\{-1, 0, 1\}$. Let P be the sum of the polynomials in S. What is the coefficient of x^4 in $P(x)$?

Answer: 70

Solution: More generally, let us compute the sum of all degree n monic polynomials whose roots are members of the set $\{-1, 0, 1\}$. Denote the sum by $P_n(x)$.

We compute

$$\begin{aligned}
P_n(x) &= \sum_{k=0}^{n} x^k \sum_{j=0}^{n-k} (x+1)^j (x-1)^{n-k-j} \\
&= \sum_{k=0}^{n} x^k (x+1)^{n-k} \frac{1-(\frac{x-1}{x+1})^{n-k+1}}{1-\frac{x-1}{x+1}} \\
&= \sum_{k=0}^{n} x^k \frac{(x+1)^{n-k+1}-(x-1)^{n-k+1}}{2} \\
&= \frac{1}{2}((x+1)^{n+1}\frac{1-(\frac{x}{x+1})^{n+1}}{1-\frac{x}{x+1}} - (x-1)^{n+1}\frac{1-(\frac{x}{x-1})^{n+1}}{1-\frac{x}{x-1}}) \\
&= \frac{1}{2}((x+1)((x+1)^{n+1}-x^{n+1}) + (x-1)((x-1)^{n+1}-x^{n+1})) \\
&= \frac{1}{2}((x+1)^{n+2} + (x-1)^{n+2} - 2x^{n+2}).
\end{aligned}$$

From the last expression, we deduce that the coefficient of x^{2k} in $P_{2m}(x)$ is $\binom{2m+2}{2k}$.

The problem asks for the coefficient of x^4 in $P_6(x)$, which is, therefore, $\binom{8}{4} = \boxed{70}$.

Problem 15 There are 300 points in space. Four planes A, B, C, and D each have the property that they split the 300 points into two equal sets. (No plane contains one of the 300 points.) What is the maximum number of points that can be found inside the tetrahedron whose faces are on A, B, C, and D?

Answer: 100

Solution: Let m be the number of points in the tetrahedron. Consider one of the planes. There are 150 points on either side of the plane. Therefore, there are $150-m$ points on the side of the plane that contains the tetrahedron which are not inside the tetrahedron. Summing over these 4 planes, we find $300 - m \leq 600 - 4m$, which simplifies to $m \leq 100$. And, in fact, we can construct a configuration of points and planes with $m = 100$ by putting 100 points in the tetrahedron and 50 points in each of the four "3-planed outer vertex" regions.

Problem 16 Let G be the set of points (x, y) such that x and y are positive integers less than or equal to 20. Say that a ray in the coordinate plane is *ocular* if it starts at $(0, 0)$ and passes through at least one point in G. Let A be the set of angle measures of acute angles formed by two distinct ocular rays. Determine

$$\min_{a \in A} \tan a.$$

Express your answer as a fraction in simplest form.

Answer: $\dfrac{1}{722}$

Solution: Replace 20 in the problem statement with n, where n is an integer greater than 4. Let (p, q) and (r, s) be two lattice points with $0 < p,\ q,\ r,\ s \leq n$ and assume that the rays from $(0, 0)$ through each point are distinct and form a minimal acute angle. (Note that for acute angle measures x and y, we have $\tan x < \tan y$ if and only if $x < y$.) Without loss of generality, assume that $r/s < p/q$. If $r/s < 1 < p/q$, then the tangent of the angle between the rays from the origin through (p, q) and $(1, 1)$ is less than that between the rays from the origin through (p, q) and (r, s). Therefore, we may assume that either both r/s and p/q are less than or equal to 1, or both are greater than or equal to 1. By symmetry, we may assume that $r/s < p/q \leq 1$. Using the trigonometric identity $\cot(\alpha - \beta) = \frac{1+\cot(\alpha)\cot(\beta)}{\cot(\beta)-\cot(\alpha)}$, we find that the cotangent of the acute angle between the rays from the origin through these two lattice points is

$$\frac{pr + qs}{ps - qr}.$$

Note that when $(p, q) = (n-1, n)$ and $(r, s) = (n-2, n-1)$, this cotangent is $2(n-1)^2$, so the maximal value of this cotangent must be at least $2(n-1)^2$. We claim that $2(n-1)^2$ is, in fact, the maximum.

To maximize this cotangent, we claim the denominator $ps - qr$ must be 1. Note that the denominator is an integer greater than or equal to 1, while the numerator, $pr + qs$, cannot exceed $2n^2$. Therefore, if the denominator is greater than 1, the largest the cotangent can be is n^2, which is less than $2(n-1)^2$ since we are assuming that $n > 3$. This implies that p and q are relatively prime since any common factor of p and q also divides $ps - qr$. Similarly, r and s must be relatively prime.

If $2(n-1)^2$ is not maximal, there must exist p, q, r, and s so that $pr + qs > 2(n-1)^2$. If all four quantities are less than n, the inequality is impossible, so at least one must be n.

Recall that p and q are relatively prime and r and s are relatively prime. Therefore, we cannot have 3 or more of the quantities p, q, r, and s be equal to n since that would force one of the pairs p and q or r and s to share the common factor n.

Since p and q are relatively prime, note that if $q > 1$ then $p < q$. Similarly, if $s > 1$, then $r < s$. (We use these facts in this paragraph and the next.) Therefore, if 2 of the quantities p, q, r, and s are equal to n, we must have $q = s = n$. Observe that the denominator $ps - qr = n(p - r)$ would then be a positive multiple of n, which contradicts the fact that the denominator must be equal to 1.

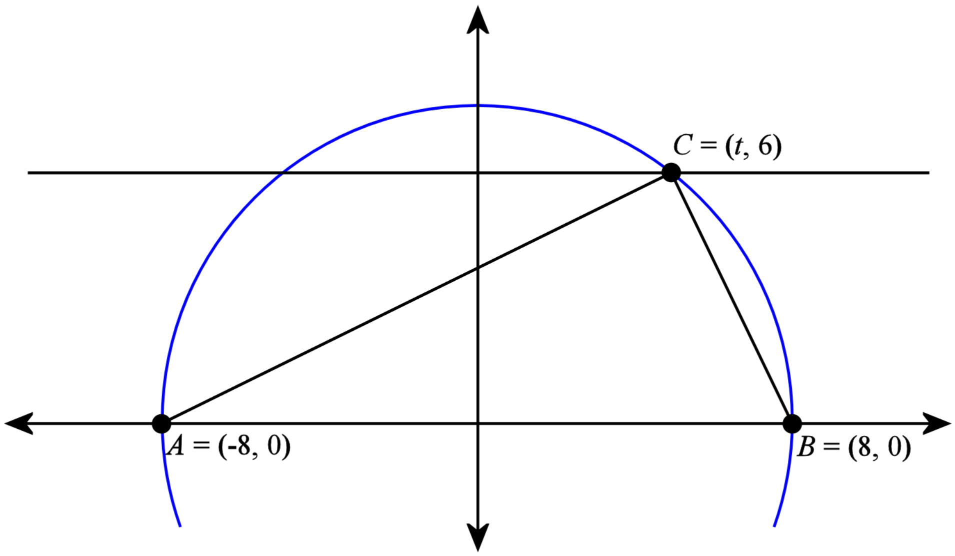
Therefore, exactly one of the quantities p, q, r, and s is equal to n. The quantity $pr + qs$ is maximized by making p, q, r, and s as large as possible. If $q = n$, then the largest s, p, and r can be are $n-1$, $n-1$, and $n-2$, respectively. With this assignment of values, $pr + qs = 2(n-1)^2$. If $s = n$, the largest q, r, and p can be are $n-1$, $n-1$, and $n-2$, respectively, which would result in $pr + qs = 2(n-1)^2$, and so with $s = n$ the cotangent cannot exceed $2(n-1)^2$. (In fact, if $s = n$, we must have $pr + qs < 2(n-1)^2$ because this assignment of values also has $r/s > p/q$, a contradiction.)

We conclude that $(p, q) = (n-1, n)$ and $(r, s) = (n-2, n-1)$ produces the maximal cotangent of $2(n-1)^2$.

When $n = 20$, we have $2(n-1)^2 = 722$, hence the answer is $\boxed{1/722}$.

Problem 17 In the coordinate plane, let $A = (-8, 0)$, $B = (8, 0)$, and $C = (t, 6)$. What is the maximum value of $\sin m\angle CAB \cdot \sin m\angle CBA$, over all real numbers t? Express your answer as a fraction in simplest form.

Answer: $\dfrac{3}{8}$



Solution: See the figure. By the law of sines, $\sin m\angle CAB \sin m\angle CBA = \frac{ab}{c}\sin^2 m\angle ACB$, where $a = BC$, $b = AC$, amd $c = AB$. Let T be the area of $\triangle ABC$. Then $T = \frac{1}{2}ab\sin m\angle ACB$. Hence, $\sin m\angle CAB \sin m\angle CBA = \frac{2T}{c^2}\sin m\angle ACB$. The only quantity that various with t in this expression is $\sin m\angle ACB$, and this is maximal when $\angle ACB$ is a right angle, and that occurs when C is on the circle of radius 8 centered at the origin. For this location of C, we have $\sin m\angle ACB = 1$, so the maximum value is $2T/c^2 = h/c = 6/16 = \boxed{3/8}$, where h is the height of $\triangle ABC$ with respect to the base $\overline{AB}$.

Problem 18 Let N be the set of square-free positive integers less than or equal to 50. (A *square-free* number is an integer that is not divisible by a perfect square bigger than 1.) How many 3-element subsets S of N are there such that the greatest common divisor of all 3 numbers in S is 1, but no pair of numbers in S is relatively prime?

Answer: 17

Solution: We denote the greatest common factor of a and b by (a, b).

We rule out possibilities so that it becomes practical to determine all solutions.

Note that every number in N is a product of distinct primes. Let T be the set of 3-element subsets S of N with the desired properties. Let

$S = \{a, b, c\} \in T$.

Note that $1 \notin S$ since the greatest common factor of any number and 1 is 1. Also, if any number in S is a prime number, then the other numbers must also be divisible by that prime, contradicting the fact the greatest common factor of all 3 numbers in S is 1. Therefore, each number in S is composite.

Let p be a prime number. We claim that if p divides any of the numbers in S, then one of the numbers in S must be greater than or equal to $3p$. For if not, then one of the numbers must be $2p$. It cannot be the case that the other two numbers are both even. Hence, one of them must also be divisible by p, which must necessarily be a higher multiple of p than $2p$, a contradiction. This fact implies that none of the numbers in S are divisible by a prime number bigger than 13, since thrice any such prime exceeds 50.

Note that (a, b), (b, c), and (c, a) must be distinct square-free integers greater than 1 which are pair-wise relatively prime, since a common factor of any two must be a common factor of all three numbers in S. Next, observe that $(a, b)(b, c)$ must divide b since (a, b) and (b, c) are relatively prime and both divide b, and hence $(a, b)(b, c) \leq 50$. The same reasoning shows that the product of any two of these greatest common factors must not exceed 50.

The foregoing considerations are enough to enable us to list all possible 3-element sets that can represent $\{(a, b), (b, c), (c, a)\}$ for a subset $\{a, b, c\} \in T$. Without loss of generality, assume $(a, b) > (b, c) > (c, a)$.

Since $(b, c) > (c, a) > 1$, we must have $(b, c) \geq 3$. Since $(a, b)(b, c) \leq 50$, we must therefore have $(a, b) < 17$.

Since (a, b), (b, c), and (c, a) are all square-free and greater than 1, they all belong to $\{2, 3, 5, 6, 7, 10, 11, 13, 14, 15\}$. And since $(a, b) > (b, c) > (c, a)$, it follows that $(a, b) \geq 5$ and $(b, c) \geq 3$.

We work through the cases.

If $(a, b) = 15$, then $(b, c) \neq 3$, for otherwise, 3 divides a, b, and c. But (b, c) cannot exceed 3 since $(a, b)(b, c) \leq 50$. Therefore, $(a, b) \neq 15$.

If $(a, b) = 14$, then $(c, a) \neq 2$, for otherwise, a, b, and c would all be even. This implies $(b, c) > 3$, but then $(a, b)(b, c) > 50$. Therefore, $(a, b) \neq 14$.

If $(a, b) = 13$, we cannot have $(b, c) > 3$ since $13 \cdot 4 = 52 > 50$. Therefore, $(b, c) = 3$ and $(c, a) = 2$.

If $(a, b) = 11$, we cannot have $(b, c) > 4$. Hence $(b, c) = 3$ and $(c, a) = 2$.

If $(a, b) = 10$, we cannot have $(b, c) > 5$, but nor can we have $(b, c) = 5$, since otherwise a, b, and c would all be divisible by 5. Hence $(b, c) = 3$ which forces $(c, a) = 2$. But then a, b, and c are all even, a contradiction. Therefore, $(a, b) \neq 10$.

Suppose $(a, b) = 7$. If $(b, c) = 6$, then (c, a) cannot be a factor of 6, and this leaves only the possibility $(c, a) = 5$. If $(b, c) = 5$, then (c, a) could be 3 or 2. If $(b, c) = 3$, then $(c, a) = 2$.

Suppose $(a, b) = 6$. Then (b, c) and (c, a) cannot be factors of 6. But there is only 1 allowable non-factor of 6 less than 6, namely, 5. Hence $(a, b) \neq 6$.

Suppose $(a, b) = 5$. Then $(b, c) = 3$ and $(c, a) = 2$.

In summary, the possible sets $\{(a, b), (b, c), (c, a)\}$ that we must examine are: $\{13, 3, 2\}$, $\{11, 3, 2\}$, $\{7, 6, 5\}$, $\{7, 5, 3\}$, $\{7, 5, 2\}$, $\{7, 3, 2\}$, and $\{5, 3, 2\}$.

For each of these possibilities, there are only a small number of possibilities for $\{a, b, c\}$, so we now compute them.

Suppose $(a, b) = 13$, $(b, c) = 3$, and $(c, a) = 2$. Then $13 \cdot 3 = 39$ divides b, and since $b \leq 50$, we must have $b = 39$. Since a must be a multiple of 13 less than 50 and not equal to 13 or 39, we must have $a = 26$. We know that $3 \cdot 2 = 6$ divides c, and c must be square-free. The only such possibilities for c are 6, 30, and 42, and all of these work.

The other cases are found in a similar manner.

We summarize the results below.

$\{(a, b), (b, c), (c, a)\}$	Corresponding sets $\{a, b, c\}$
$\{13, 3, 2\}$	$\{26, 39, 42\}, \{26, 39, 30\}, \{26, 39, 6\}$
$\{11, 3, 2\}$	$\{22, 33, 42\}, \{22, 33, 30\}, \{22, 33, 6\}$
$\{7, 6, 5\}$	$\{35, 42, 30\}$
$\{7, 5, 3\}$	$\{42, 35, 15\}, \{21, 35, 30\}, \{21, 35, 15\}$
$\{7, 5, 2\}$	$\{42, 35, 10\}, \{14, 35, 30\}, \{14, 35, 10\}$
$\{7, 3, 2\}$	$\{14, 21, 30\}, \{14, 21, 6\}$
$\{5, 3, 2\}$	$\{10, 15, 42\}, \{10, 15, 6\}$

In conclusion, we find $\boxed{17}$ possibilities in total.

Problem 19 Let T be a regular tetrahedron. Let t be the regular tetrahedron whose vertices are the centers of the faces of T. Let O be the circumcenter of either tetrahedron. Given a point P different from O, let $m(P)$ be the midpoint of the points of intersection of the ray $\overrightarrow{OP}$ with t and T. Let S be the set of eight points $m(P)$ where P is a vertex of either t or T. What is the volume of the convex hull of S divided by the volume of t? Express your answer as a fraction in simplest form.

Answer: $\dfrac{25}{3}$

Solution: Place the situation in a coordinate system with origin at O. We shall conflate the terms "point" and "position vector" throughout this solution. The reader may find it helpful to refer to the figure.

Let T_1, T_2, T_3, and T_4 be the four position vectors pointing from O to each of the vertices of T. By symmetry, $T_1 + T_2 + T_3 + T_4 = 0$.

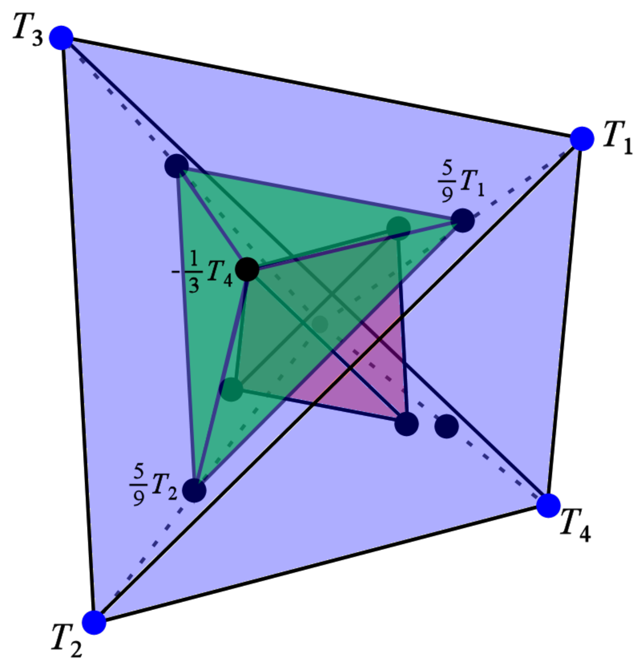
The vertices of t are the averages of any subset of 3 of the 4 vectors T_k. Since $T_1 + T_2 + T_3 + T_4 = 0$, we see that the vertices of t are given by $-T_k/3$. We note that this shows that t is a regular tetrahedron whose scale factor compared to T is $1/3$.

We now compute $m(T)$ where T is a vertex of either t or T. Since $-T_k/3$ is both a vertex of t and the center of a face of T, we have $m(-T_k/3) = -T_k/3$. To compute $m(T_k)$, we need to know where the ray $\overrightarrow{OT_k}$ intersects both t and T. Since T_k is a vertex of T, the ray $\overrightarrow{OT_k}$ intersects T at T_k. On the other hand, the center of the face of t opposite its vertex $-T_k/3$ is given by $\frac{1}{3}\sum_{i\neq k}(-T_i/3) = T_k/9$. Thus, $m(T_k) = (T_k + T_k/9)/2 = 5T_k/9$. Hence, the set S consists of the points $-T_k/3$ and $5T_k/9$, $k = 1, 2, 3, 4$.

Let M be the regular tetrahedron whose vertices are the 4 points $5T_k/9$. (It is T scaled down by a factor of $5/9$.) The centroids of the faces of M are given by $-5T_k/27$. Because $-T_k/3 = \frac{9}{5}(-5T_k/27)$, the points $-T_k/3$ are external to M.

Consider the solid V obtained by placing a triangular pyramid upon each face of M. Specifically, upon the face of M that has centroid $-5T_k/27$, place the triangular pyramid with base that face and apex the point $-T_k/3$. Because V is a union of tetrahedra whose vertices belong to S, the convex hull of S must contain V.

We claim that V is convex, and is therefore the convex hull of S. To show this, we show that V lies entirely on one-side of each of the planes that contain its faces. (Convexity follows since this would show that V is an intersection of convex half-spaces, and the intersection of convex sets is convex.) By symmetry, it suffices to check this for just one of the planes that contains a face, which we shall take to be the face with vertices $5T_1/9$, $5T_2/9$, and $-T_4/3$. The reader may verify that the (closed) half-space H defined by the plane containing this face and O consists of the points $a(5T_1/9) + b(5T_2/9) + c(-T_4/3)$, where a, b, and c are real numbers such that $a + b + c \leq 1$. Define



The tetrahedra T and t, as well as one of the pyramids (in green) placed upon a face of M to form V.

$\langle a, b, c\rangle \equiv a(5T_1/9) + b(5T_2/9) + c(-T_4/3)$. The points in S are:

P		$\langle a,$	$b,$	$c\rangle$	$a+b+c$	$P \in H$?
$5T_1/9$	$=$	$\langle 1,$	$0,$	$0\rangle$	1	yes
$5T_2/9$	$=$	$\langle 0,$	$1,$	$0\rangle$	1	yes
$5T_3/9$	$=$	$\langle -1,$	$-1,$	$5/3\rangle$	$-1/3$	yes
$5T_4/9$	$=$	$\langle 0,$	$0,$	$-5/3\rangle$	$-5/3$	yes
$-T_1/3$	$=$	$\langle -3/5,$	$0,$	$0\rangle$	$-3/5$	yes
$-T_2/3$	$=$	$\langle 0,$	$-3/5,$	$0\rangle$	$-3/5$	yes
$-T_3/3$	$=$	$\langle 3/5,$	$3/5,$	$-1\rangle$	$1/5$	yes
$-T_4/3$	$=$	$\langle 0,$	$0,$	$1\rangle$	1	yes

which shows that $V \subset H$.

We must now compute the ratio of the volume of V to the volume of t. Since $-T_k/3 = (9/5)(-5T_k/27)$, the addition of the pyramids on each face of M to form V increases the volume of V by a factor of 9/5. On the other hand, the scale factor of M to t is 5/3 (because the scale factor of M to T is 5/9 and the scale factor of T to t is 3). Therefore, the ratio of the volume of V to that of t is $(9/5)(5/3)^3 = \boxed{25/3}$.

Note: The convex hull of S is called a "triakis tetrahedron."

Problem 20 Let G be the set of points (x, y) such that x and y are positive integers less than or equal to 6. A *magic grid* is an assignment of an integer to each point in G such that, for every square with horizontal and vertical sides and all four vertices in G, the sum of the integers assigned to the four vertices is the same as the corresponding sum for any other such square. A magic grid is formed so that the product of all 36 integers is the smallest possible value greater than 1. What is this product?

Answer: 6561

Solution: Consider first a 3 by 3 square grid subset of the magic grid and assign variables to the grid points as indicated below.

$$\begin{array}{ccc} A & B & X \\ C & D & U_1 \\ U_2 & U_3 & U_4 \end{array}$$

We claim that the U_k are determined by A, B, C, D, and X. The upper right 2 by 2 grid tells us that $U_1 = A+C-X$. We must also have $U_2+U_3 = A+B$, $U_2 + U_4 = B + C + D - X$, and $U_3 + U_4 = A + B + C - U_1$. This system of linear equations in the unknowns U_2, U_3, and U_4 can be solved by adding two of the equations while subtracting the third, in all 3 possible ways. We find the following:

$$\begin{array}{ccc} A & B & X \\ \\ C & D & A+C-X \\ \\ S/2-X & \frac{A+B-C-D}{2}+X & S/2-A \end{array}$$

where $S = A + B + C + D$. Notice that the sum of the entries assigned to opposite corners turns out to be $S/2$. In other words, the opposite corners of any 3 by 3 square subgrid of a magic grid must be assigned numbers that sum to $S/2$. This implies that the four corners of any 5 by 5 square subgrid of a magic grid must be assigned the same number. If we now think of our 3 by 3 subgrid as occupying the upper left 3 by 3 corner of the 6 by 6 magic grid, we conclude that $A = B = C = D = S/4$, since the points that A, B, C, and D are assigned to in the top left 2 by 2 corner are each corners of a 5 by 5 subgrid within the 6 by 6 magic grid.

Replacing B, C, and D with A, we now know the following:

$$
\begin{array}{cccccc}
A & A & X & ? & A & A \\
A & A & 2A-X & ? & A & A \\
2A-X & X & A & ? & ? & ? \\
? & ? & ? & ? & ? & ? \\
A & A & ? & ? & A & A \\
A & A & ? & ? & A & A
\end{array}
$$

We then apply the fact that points in opposite corners of a 3 by 3 subgrid must be assigned numbers that sum to $2A = S/2$ to obtain:

$$
\begin{array}{cccccc}
A & A & X & 2A-X & A & A \\
A & A & 2A-X & ? & A & A \\
2A-X & X & A & A & 2A-X & X \\
X & ? & A & A & X & ? \\
A & A & X & 2A-X & A & A \\
A & A & 2A-X & ? & A & A
\end{array}
$$

The remaining unassigned points are each corners of squares where all the other corners have been assigned values, and we use this to determine the 4 remaining values.

$$\begin{array}{cccccc}
A & A & X & 2A-X & A & A \\
A & A & 2A-X & X & A & A \\
2A-X & X & A & A & 2A-X & X \\
X & 2A-X & A & A & X & 2A-X \\
A & A & X & 2A-X & A & A \\
A & A & 2A-X & X & A & A
\end{array}$$

We leave it to the reader to verify that the above assignment of values does result in a magic grid.

The product of all the entries is $A^{20}X^8(2A-X)^8$. This product must be greater than 1, so $A \neq 0$, $X \neq 0$, and $2A - X \neq 0$, therefore $|A| \geq 1$, $|X| \geq 1$, and $|2A - X| \geq 1$.

If $A = 1$, consider $f(X) = |X||2 - X|$. Since the product cannot be 0, the value of X cannot be 0 or 2, since $f(0) = f(2) = 0$. Since the product cannot be 1, the value of X cannot be 1, since $f(1) = 1$. If $X \geq 3$, then $f(X) \geq |X| \geq 3$. If $X \leq -1$, then $f(X) \geq |2 - X| \geq 3$. So the product is at least 3^8, and when $X = -1$ or $X = 3$, it is exactly 3^8.

If $A = -1$, we can multiply all 36 entries by -1 and the new magic grid would satisfy all required conditions, and the product of all 36 entries in the new magic grid would be the same as in the original magic grid. By case $A = 1$, this product is at least 3^8.

If $|A| \geq 2$, then $A^{20}X^8(2A - X)^8 \geq 2^{20} > 3^8$.

We conclude that the answer is $3^8 = \boxed{6561}$.