# Math Prize for Girls Olympiad

**Advantage Testing Foundation/Jane Street**

**Time Limit: 4 hours**

**Friday, December 2, 2022**

1. Let $a$, $b$, $c$ be positive integers with $a \leq 10$. Suppose the parabola $y = ax^2 + bx + c$ meets the $x$-axis at two distinct points $A$ and $B$. Given that the length of $\overline{AB}$ is irrational, determine, with proof, the smallest possible value of this length, across all such choices of $(a, b, c)$.

2. Determine, with proof, whether or not there exists a *non-isosceles* trapezoid $ABCD$ such that the lengths $AC$ and $BD$ both lie in the set $\{DA + AB, AB + BC, BC + CD, CD + DA, AB + CD, BC + DA\}$.

3. Serena has written 20 copies of the number 1 on a board. In a move, she is allowed to

   - erase two of the numbers and replace them with their sum, or
   - erase one number and replace it with its reciprocal.

   Whenever a fraction appears on the board, Serena writes it in simplest form. Prove that Serena can never write a fraction less than 1 whose numerator is over 9000, regardless of the number of moves she makes.

4. Let $n > 1$ be an integer. Let $A$ denote the set of divisors of $n$ that are less than $\sqrt{n}$. Let $B$ denote the set of divisors of $n$ that are greater than $\sqrt{n}$. Prove that there exists a bijective function $f \colon A \to B$ such that $a$ divides $f(a)$ for all $a \in A$.

   (We say $f$ is *bijective* if for every $b \in B$ there exists a unique $a \in A$ with $f(a) = b$.)