Solutions to Math Prize Olympiad 2022

Advantage Testing Foundation / Jane Street

December 2, 2022

§1 Problem 1

The answer is $\frac{\sqrt{13}}{9}$, achieved with the quadratic $9x^2 + 7x + 1$.

First solution Let $D = b^2 - 4ac$. By the quadratic formula, the two roots are given by

$$\begin{aligned} r_1 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ r_2 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ \implies AB = r_1 - r_2 &= \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{\sqrt{D}}{2a}. \end{aligned}$$

At this point, we note that $(a, b, c) = (9, 7, 1)$ gives $D = 13$, and hence the bound we claimed is achievable.

We now proceed to show $\frac{\sqrt{13}}{9}$ is best possible. In general we have $D \equiv 0 \pmod 4$ or $D \equiv 1 \pmod 4$, and D is not a perfect square. So it suffices to analyze five cases: $D = 5$, $D = 8$, $D = 12$, $D = 13$, and $D \geq 17$.

- In the first case, if $D = 5$, then it follows that $b^2 \equiv 5 \pmod{4a}$.

  Since 5 is not a square modulo 8 (the squares are 0, 1, 4), it follows a is odd.

  Since 5 is not a square modulo 3 (the squares are 0 and 1), it follows $3 \nmid a$.

  Since 5 is not a square modulo 7 (the squares are 0, 1, 2, 4), it follows $7 \nmid a$.

  This means if $D = 5$ then $a \leq 5$. Hence $AB \geq \frac{\sqrt{5}}{5} > \frac{\sqrt{13}}{9}$ and this case is completed.

- Now suppose $D = 8$, meaning $b^2 \equiv 8 \pmod{4a}$.

  Since 8 is not a square modulo 5 (the squares are 0, 1, 4), it follows $5 \nmid a$.

  Since 8 is not a square modulo 3 (the squares are 0, 1), it follows $3 \nmid a$.

  Since 8 is not a square modulo 16 (the squares are 0, 1, 4, 9), it follows $4 \nmid a$.

  This means if $D = 8$ then $a \leq 7$. Hence $AB \geq \frac{\sqrt{8}}{7} > \frac{\sqrt{13}}{9}$ and this case is completed.

- Now suppose $D = 12$, meaning $b^2 \equiv 12 \pmod{4a}$.

  Since 12 is not a square modulo 5 (the squares are 0, 1, 4), it follows $5 \nmid a$.

  Since 12 is not a square modulo 9 (the squares are 0, 1, 4, 7), it follows $3 \nmid a$.

  This means if $D = 12$ then $a \leq 8$. Hence $AB \geq \frac{\sqrt{12}}{8} > \frac{\sqrt{13}}{9}$ and this case is completed.

- Now suppose $D = 13$, meaning $b^2 \equiv 13 \pmod{4a}$. Since 13 is not a square modulo 8, it follows a is odd. Hence in this case $AB \ge \frac{\sqrt{13}}{9}$, which is what we wanted.

- Finally, if $D \ge 17$, we have the bound $AB \ge \frac{\sqrt{D}}{10} \ge \frac{\sqrt{17}}{10} > \frac{\sqrt{13}}{9}$.

Second solution Alternatively, rather than by cases on D, one may also use cases on the value of a. For a fixed value of a, the value $D = b^2 - 4ac$ takes values equal to the quadratic residues modulo $4a$. So, one may search the non-square $D \in \{2, 3, 5, 6, 7, 8, 10, 11, \dots\}$ for the first value which could occur. (In fact, since the squares modulo 4 are 0 and 1, we may restrict to those values of D which are not perfect squares but are $0 \pmod 4$ or $1 \pmod 4$; that is $D \in \{5, 8, 12, 13, 17, 20, 21, \dots\}$.) These values for $a \ge 6$ are listed below:

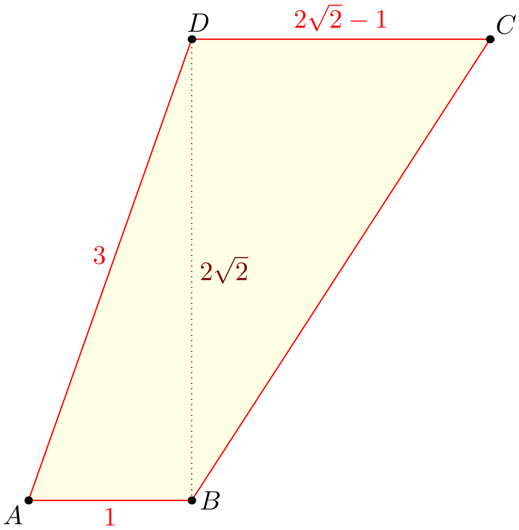
a	$D_{\min}$	$\sqrt{D}/a$
$a = 6$	$D = 12$	$\sqrt{12}/6 \approx 0.577$
$a = 7$	$D = 8$	$\sqrt{8}/7 \approx 0.404$
$a = 8$	$D = 17$	$\sqrt{17}/8 \approx 0.515$
$a = 9$	$D = 13$	$\sqrt{13}/9 \approx 0.401$
$a = 10$	$D = 20$	$\sqrt{20}/10 \approx 0.447$

In this table, $\sqrt{13}/9$ is the smallest value occurring. And for $a \le 5$, since $D \ge 5$ we would always get a value at least $\sqrt{5}/5 > \sqrt{13}/9$ anyways. This concludes the proof.

§2 Problem 2

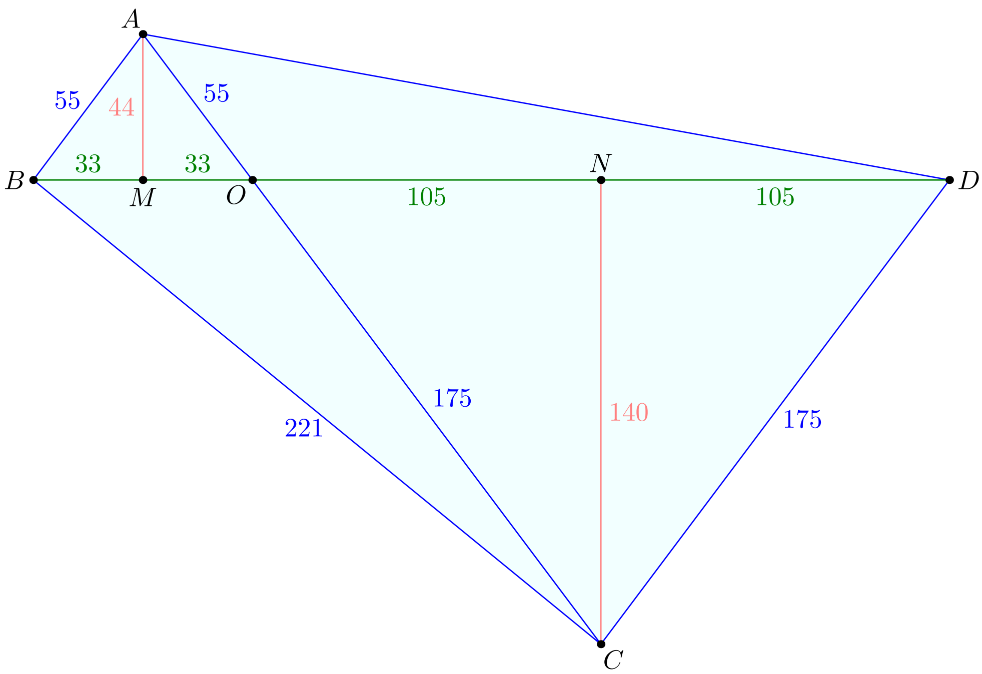
The answer is yes, such a trapezoid exists. We present two possible direct constructions and one indirect one.

First construction Let $A = (0, 0)$, $B = (1, 0)$, $C = (2\sqrt{2}, 2\sqrt{2})$, $D = (1, 2\sqrt{2})$, as shown below.



Then $AC = 4 = 1 + 3 = AB + AD$, and $BD = 2\sqrt{2} = 1 + (2\sqrt{2} - 1) = AB + CD$. Also, we clearly have $AB \parallel CD$ (since $\angle DBA = \angle CDB = 90^\circ$) and $3 = AD \neq BC$, so this construction is valid.

Second construction Construct two similar right triangles AOM and CON, where $OM = 33$, $MA = 44$, $OA = 55$ and $ON = 105$, $NC = 140$, $CO = 175$. Situate these triangles such that AOC and MON are collinear. Finally, let B and D be the reflections of O over M and N, respectively. The resulting figure is depicted below.



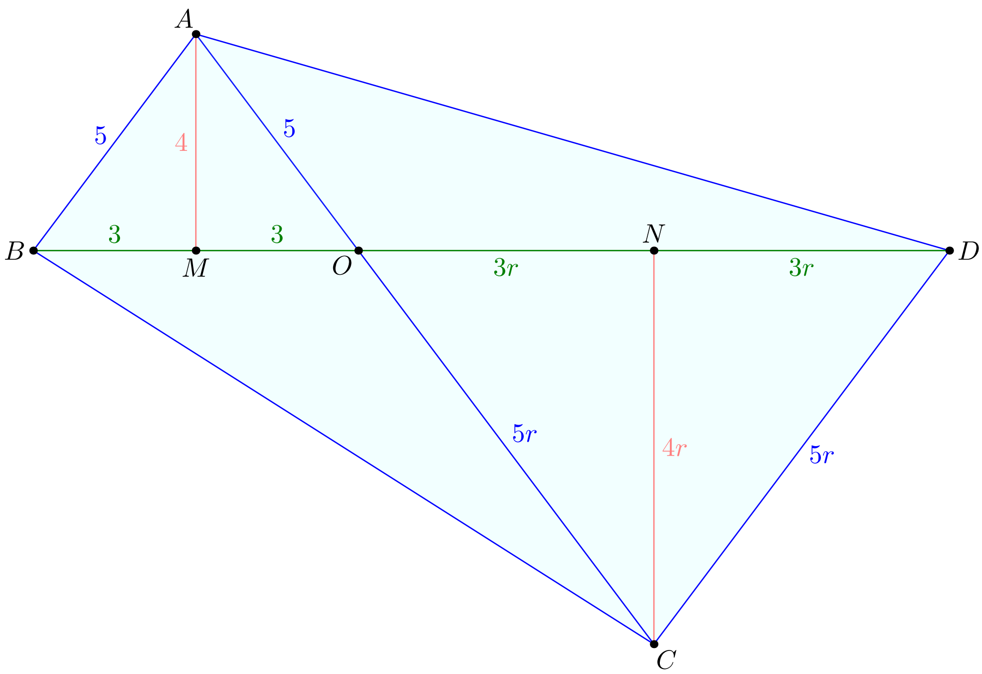
Then because $\triangle AOB$ and $\triangle COD$ are similar, it follows that AB and CD are parallel. In that case, we have

$$\begin{aligned} AB &= AO = 55 \\ CD &= CO = 175 \\ BC &= \sqrt{BN^2 + NC^2} = \sqrt{171^2 + 140^2} = 221 \\ AC &= AO + CO = 55 + 175 = 230 = AB + CD \\ BD &= BO + DO = 2(33 + 105) = 276 = BC + CD. \end{aligned}$$

(The length of AD is not relevant for this solution.) This completes the construction.

> **Remark.** The numbers selected here may seem magical in nature. Really, the underlying idea is to construct two similar isosceles triangles AOB and COD as above, so that $AB \parallel CD$ and $AC = AB + CD$ are automatically true. In that case, the only condition that needs to hold is for $BD = AB + BC$ to be true. Because we have a choice of three numbers (the length AO, BO, CO determine the figure), it should be possible to make this equation true, and one simply needs to exhibit one solution. The lengths here were chosen after some calculation to yield a construction in which all lengths were integers, but this is neither necessary nor important for the solution to work.

Indirect construction using continuity We develop the ideas mentioned in the preceding remark by showing how one can indirectly prove the existence of a valid trapezoid, without having to actually find all the necessary constants. Indeed, we again construct two similar right triangles AOM and CON, but this time we set where $OM = 3$, $MA = 4$, $OA = 5$ (say) and $ON = 3r$, $NC = 4r$, $CO = 5r$, for some $r > 0$. Then let B and D be the reflections of O over M and N, respectively.



Because $BD = 6 + 6r$ and $AC = 5 + 5r$, we have $BD > AC$, and this trapezoid is not isosceles for any value of r. Now, we vary the parameter r and consider the function

$$f(r) := AB + BC - BD = 5 + \sqrt{(3r+6)^2 + (4r)^2} - (6 + 6r).$$

Note that this function is continuous and

$$f(0.001) = 5 + \sqrt{3.006^2 + 4^2} - 6.006 > 0$$
$$f(1000) = 5 + \sqrt{3006^2 + 4000^2} - 6006 < 0.$$

so by the intermediate value theorem, there must be some $0.001 < r < 1000$ for which $f(r) = 0$. That value of r gives a valid construction.

Remark. As we saw in the previous solution, $r = \frac{35}{11}$ works. In fact, it is the unique value of r for which $f(r) = 0$.

The choice of a 3-4-5 triangle in this construction is just for concreteness; many other dimensions would work as well.

§3 Problem 3

Let $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, ... denote the sequence of Fibonacci numbers. We need one classical lemma before we proceed.

Lemma 3.1
For any integers i and j, $F_i F_j + F_{i+1} F_{j+1} = F_{i+j+1}$.

Proof, only for completeness. This lemma can be proved by using the explicit *Binet's formula* for the Fibonacci numbers. Let $\varphi = \frac{1}{2}(1+\sqrt{5})$, so we have the formula

$$F_n = \frac{1}{\sqrt{5}}\left(\varphi^n - (-\varphi)^{-n}\right)$$

holds identically. In this situation,

$$\begin{aligned} F_i F_j + F_{i+1} F_{j+1} &= \frac{1}{5}\Big[\left(\varphi^i - (-\varphi)^{-i}\right)\left(\varphi^j - (-\varphi)^{-j}\right) \\ &\qquad + \left(\varphi^{i+1} - (-\varphi)^{-(i+1)}\right)\left(\varphi^{j+1} - (-\varphi)^{-(j+1)}\right)\Big] \\ &= \frac{1}{5}\left[\varphi^{i+j} + (-\varphi)^{-(i+j)} + \varphi^{i+j+2} + (-\varphi)^{-(i+j+2)}\right] = F_{i+j+1} \end{aligned}$$

where the last line uses the fact that $\varphi^2 + 1 = \sqrt{5}\varphi$. □

The idea is to show by induction the following lemma:

Claim — Given n starting 1's, any number we achieve has the property that

- the larger of the numerator and denominator is at most F_{n+1},
- the sum of the two is at most F_{n+2}.

Proof of main claim. We'll be looking at the situation where Serena is merging two fractions

$$\frac{a}{b} + \frac{p}{q} = \frac{aq+pb}{bq}$$

where the left fraction came from k starting 1's and the right fraction came from ℓ starting 1's. In order to complete the induction, it is enough to prove the following.

Lemma
Suppose that

$$\begin{cases} \max(a,b) & \le F_{k+1} \\ a+b & \le F_{k+2} \end{cases} \quad \text{and} \quad \begin{cases} \max(p,q) & \le F_{\ell+1} \\ p+q & \le F_{\ell+2}. \end{cases}$$

Then it follows that

$$aq + pb \le F_{k+\ell+1} \qquad \text{and} \qquad aq + pb + bq \le F_{k+\ell+2}.$$

Proof of lemma. Without loss of generality suppose $q \geq p$. For the first inequality,

$$\begin{aligned} aq + bp &= a \cdot (q-p) + (a+b) \cdot p \\ &\leq F_{k+1}(q-p) + F_{k+2} \cdot p \\ &= F_{k+1} \cdot q + F_k \cdot p \\ &= (F_{k+1} - F_k) \cdot q + F_k \cdot (p+q) \\ &\leq F_{k-1}F_{\ell+1} + F_k F_{\ell+2} = F_{k+\ell+1}. \end{aligned}$$

For the second inequality, we have

$$\begin{aligned} aq + bp + bq &\leq a \cdot F_{\ell+1} + b \cdot F_{\ell+2} \\ &= b \cdot F_\ell + (a+b) \cdot F_{\ell+1} \\ &\leq F_{k+1}F_\ell + F_{k+2}F_{\ell+1} = F_{k+\ell+2}. \end{aligned} \qquad \blacksquare$$

The claim now follows directly by induction on n. In the penultimate step, we have two fractions $\frac{a}{b}$ and $\frac{p}{q}$ which were created from k ones and ℓ ones respectively, where $k + \ell = n$. Since $aq + pb$ is the new numerator and $aq + bp + bq$ is the new denominator before any simplification, we're done. $\square$

To finish, compute the first several Fibonacci numbers:

$$\begin{array}{lllll} F_1 = 1, & F_2 = 1, & F_3 = 2, & F_4 = 3, & F_5 = 5 \\ F_6 = 8, & F_7 = 13, & F_8 = 21, & F_9 = 34, & F_{10} = 55 \\ F_{11} = 89, & F_{12} = 144, & F_{13} = 233, & F_{14} = 377, & F_{15} = 610 \\ F_{16} = 987, & F_{17} = 1597, & F_{18} = 2584, & F_{19} = 4181, & F_{20} = 6765 \\ F_{21} = 10946, & F_{22} = 17711. & & & \end{array}$$

If Serena's final fraction is $\frac{a}{b}$ for $a < b$ then we should have

$$a < \frac{a+b}{2} \leq \frac{17711}{2} < 9000$$

as desired.

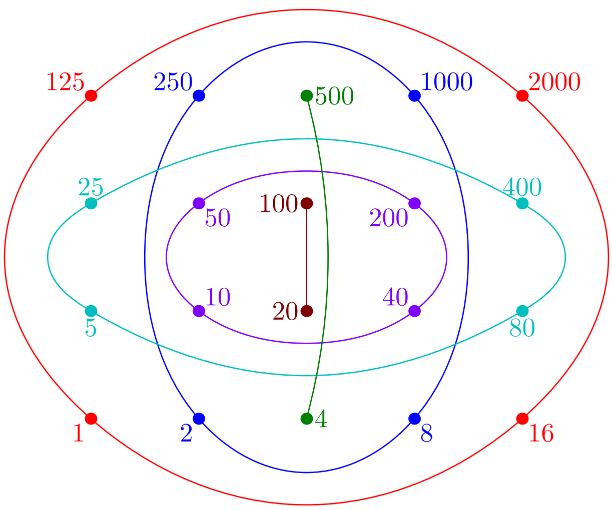
§4 Problem 4

Setup In the following solution, $\nu_p(n)$ denotes the exponent of the prime p in the prime factorization of n.

We define two divisors d and d' of n to be *related* if, for every prime p dividing n, we have either

$$\nu_p(d) + \nu_p(d') = \nu_p(n) \quad \text{or} \quad \nu_p(d) = \nu_p(d').$$

This is an *equivalence relation*, so it partitions the divisors of n into *equivalence classes*. See the illustration below, which shows the classes for $n = 2^4 \cdot 5^3 = 2000$. The equivalence classes are the points of the same color connected by curves.



Fix any such equivalence class C. We will show one can construct a bijection

$$f_C : A \cap C \to B \cap C$$

that has the desired divisibility property $d \mid f_C(d)$. This will solve the problem.

Main claim The following claim is the heart of the problem:

Claim — Let X be a finite set, and $\mathcal{F}$ be a family of subsets of X. Assume that for every set $A \in \mathcal{F}$, all the subsets of A are also in $\mathcal{F}$.

Then there exist a bijection $\sigma : \mathcal{F} \to \mathcal{F}$ satisfying $\sigma(A) \cap A = \varnothing$ for every $A \in \mathcal{F}$.

Proof. We proceed by induction on $|X|$, with the base case $|X| = 0$ being easy. For the inductive step, let $x \in X$ be any element. We can split the sets of $\mathcal{F}$ in three ways:

- Sets containing x, which we call *small*;
- Sets $S \in \mathcal{F}$ which don't contain x, but for which $S \cup \{x\} \in \mathcal{F}$, which we call *big*;
- All remaining sets, which we call *neutral*.

There is a natural bijection from small to big sets by $S \mapsto S \cup \{x\}$. Also, by the inductive hypothesis, there is a bijection

$$\sigma_{\text{small}} : \{\text{small sets}\} \to \{\text{small sets}\}$$
$$\sigma_{\text{neutral}} : \{\text{neutral sets}\} \to \{\text{neutral sets}\}$$

with the desired property. Then we can define our desired σ on all small, large, neutral sets respectively by

$$\begin{aligned} \sigma(S) &= \sigma_{\text{small}}(S) \cup \{x\} && \text{for all small } S \\ \sigma(S \cup \{x\}) &= \sigma_{\text{small}}(S) && \text{for all small } S \\ \sigma(T) &= \sigma_{\text{neutral}}(T) && \text{for neutral } T. \end{aligned}$$

This works and completes the induction. □

> **Remark.** This claim apparently appeared previously on an Iranian olympiad in 2000, see https://aops.com/community/p20560.
>
> It is possible to phrase this using Hall's marriage lemma. However, as far as I am aware, doing this *still* requires an induction similar to the one specified above, in which one deletes an element $x \in X$. So we chose to just present the direct solution which did not cite the marriage lemma.
>
> It's also possible to modify the induction in such a way that π is actually an *involution*, meaning $\pi(\pi(A)) = A$ for all A. This uses the same induction, but the collation step is more complicated; see the above link for details.

Completion For each divisor d in C, we define

$$S(d) = \left\{ p \text{ prime} \mid \nu_p(d) > \frac{1}{2}\nu_p(n) \right\}.$$

We invoke the claim where X is the set of prime divisors of n and $\mathcal{F} = \{S(d) \mid d \in A \cap C\}$. This gives us a bijection $\sigma \colon A \cap C \to A \cap C$ so that $S(d) \cap S(\sigma(d)) = \varnothing$ for each $d \in A \cap C$. Then we define

$$f_C(d) = \frac{n}{\sigma(d)} \in B \cap C$$

and this gives the desired f_C, by construction.