The Fourteenth Annual Math Prize for Girls

Sunday, October 9, 2022

Version A

DIRECTIONS

1. Do not open this test until your proctor instructs you to.
2. Fill out the top of your answer sheet. Your student number is on your name tag. Your test version (A, B, C, or D) is above on this page.
3. You may use pencils, pens, and erasers. You may also use the scratch paper that we provide. You may NOT use calculators, rulers, protractors, compasses, graph paper, or books.
4. Write your final answers on the answer sheet. When a problem specifies a particular form for the answer, write your answer in that form. Write your answers clearly. We will collect only the answer sheet from you.

NOTES

1. This test contains 20 problems. You will have 150 minutes (2.5 hours) to take the test. Your score will be the number of correct answers.
2. Figures are not necessarily drawn to scale.
3. Good luck!

Math Prize for Girls 2022

1. Determine the real value of t that minimizes the expression

$$\sqrt{t^2 + (t^2 - 1)^2} + \sqrt{(t - 14)^2 + (t^2 - 46)^2}.$$

Express your answer as a fraction in simplest form.

2. Let b and c be random integers from the set $\{1, 2, \ldots, 100\}$, chosen uniformly and independently. What is the probability that the roots of the quadratic $x^2 + bx + c$ are real? Express your answer as a fraction in simplest form.

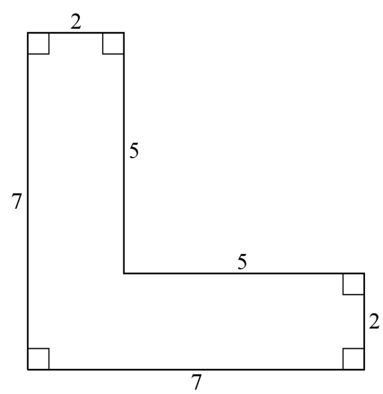
3. Let $ABCD$ be the square face of a cube with edge length 2. A plane P that contains A and the midpoint of $\overline{BC}$ splits the cube into two pieces of the same volume. What is the square of the area of the intersection of P and the cube?

4. Determine the largest integer n such that $n < 103$ and $n^3 - 1$ is divisible by 103.

5. Given a real number a, the *floor* of a, written $\lfloor a \rfloor$, is the greatest integer less than or equal to a. For how many real numbers x such that $1 \le x \le 20$ is

$$x^2 + \lfloor 2x \rfloor = \lfloor x^2 \rfloor + 2x\,?$$

6. An L-shaped region is formed by attaching two 2 by 5 rectangles to adjacent sides of a 2 by 2 square as shown below.



The resulting shape has an area of 24 square units. How many ways are there to tile this shape with 2 by 1 dominos (each of which may be placed horizontally or vertically)?

7. The quadrilateral $ABCD$ is an isosceles trapezoid with $AB = CD = 1$, $BC = 2$, and $DA = 1+\sqrt{3}$. What is the measure of $\angle ACD$ in degrees?

8. Let S be the set of numbers of the form $n^5 - 5n^3 + 4n$, where n is an integer that is not a multiple of 3. What is the largest integer that is a divisor of every number in S?

9. Let $\triangle PQO$ be the unique right isosceles triangle inscribed in the parabola $y = 12x^2$ with P in the first quadrant, right angle at Q in the second quadrant, and O at the vertex $(0, 0)$. Let $\triangle ABV$ be the unique right isosceles triangle inscribed in the parabola $y = x^2/5 + 1$ with A in the first quadrant, right angle at B in the second quadrant, and V at the vertex $(0, 1)$. The y-coordinate of A can be uniquely written as $uq^2 + vq + w$, where q is the x-coordinate of Q and u, v, and w are integers. Determine $u + v + w$.

10. An algal cell population is found to have a_k cells on day k. Each day, the number of cells at least doubles. If $a_0 \geq 1$ and $a_3 \leq 60$, how many quadruples of integers (a_0, a_1, a_2, a_3) could represent the algal cell population size on the first 4 days?

11. Let A, B, C, D, E, and F be 6 points around a circle, listed in clockwise order. We have $AB = 3\sqrt{2}$, $BC = 3\sqrt{3}$, $CD = 6\sqrt{6}$, $DE = 4\sqrt{2}$, and $EF = 5\sqrt{2}$. Given that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent, determine the square of AF.

12. Solve the equation

$$\sin 9^\circ \sin 21^\circ \sin(102^\circ + x) = \sin 30^\circ \sin 42^\circ \sin x,$$

for x where x is a degree measure between 0° and 90°.

13. The roots of the polynomial $x^4 - 4ix^3 + 3x^2 - 14ix - 44$ form the vertices of a parallelogram in the complex plane. What is the area of the parallelogram?

14. Across the face of a rectangular post-it note, you idly draw lines that are parallel to its edges. Each time you draw a line, there is a 50% chance it'll be each direction and you never draw over an existing line or the edge of the post-it note. After a few minutes, you notice that you've drawn 20 lines. What is the expected number of rectangles that the post-it note will be partitioned into?

15. What is the smallest positive integer m such that $15!m$ can be expressed in more than one way as a product of 16 distinct positive integers, up to order?

16. A snail begins a journey starting at the origin of a coordinate plane. The snail moves along line segments of length $\sqrt{10}$ and in any direction such that the horizontal and vertical displacements are both integers. As the snail moves, it leaves a trail tracing out its entire journey. After a while, this trail can form various polygons. What is the smallest possible area of a polygon that could be created by the snail's trail? Express your answer as a fraction in simplest form.

17. Let O be the set of odd numbers between 0 and 100. Let T be the set of subsets of O of size 25. For any finite subset of integers S, let $P(S)$ be the product of the elements of S. Define $n = \sum_{S \in T} P(S)$. If you divide n by 17, what is the remainder? (The remainder is an integer between 0 and 16, inclusive.)

18. Let A be the locus of points (α, β, γ) in the $\alpha\beta\gamma$-coordinate space that satisfy the following properties:

I We have α, β, $\gamma > 0$.

II We have $\alpha + \beta + \gamma = \pi$.

III The intersection of the three cylinders in the xyz-coordinate space given by the equations

$$\begin{aligned} y^2 + z^2 &= \sin^2 \alpha \\ z^2 + x^2 &= \sin^2 \beta \\ x^2 + y^2 &= \sin^2 \gamma \end{aligned}$$

is nonempty.

Determine the area of A. Express your answer in terms of π.

19. Let S_- be the semicircular arc defined by

$$(x+1)^2 + (y - \frac{3}{2})^2 = \frac{1}{4} \text{ and } x \leq -1.$$

Let S_+ be the semicircular arc defined by

$$(x-1)^2 + (y - \frac{3}{2})^2 = \frac{1}{4} \text{ and } x \geq 1.$$

Let R be the locus of points P such that P is the intersection of two lines, one of the form $Ax + By = 1$ where $(A, B) \in S_-$ and the other of the form $Cx + Dy = 1$ where $(C, D) \in S_+$. What is the area of R? Express your answer as a fraction in simplest form.

20. Let $a_n = n(2n + 1)$. Evaluate

$$\left| \sum_{1 \leq j < k \leq 36} \sin(\frac{\pi}{6}(a_k - a_j)) \right|.$$