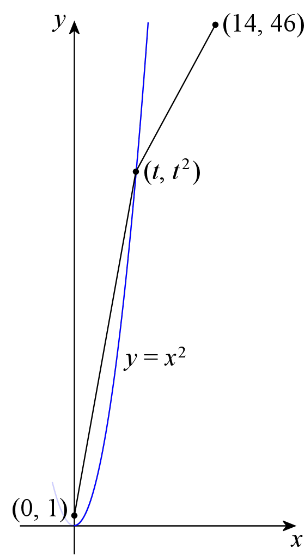
Math Prize for Girls
2022 Solutions

Problem 1 Determine the real value of t that minimizes the expression
$$\sqrt{t^2 + (t^2 - 1)^2} + \sqrt{(t - 14)^2 + (t^2 - 46)^2}.$$
Express your answer as a fraction in simplest form.

Answer: $\dfrac{7}{2}$



Solution: Consider the two-legged path from $(0, 1)$ to (t, t^2) to $(14, 46)$. The expression gives the length of this path, which would be minimized if (t, t^2) is on the line segment connecting the two points $(0, 1)$ and $(14, 46)$. Thus, we seek t such that $0 < t < 14$ and
$$\frac{t^2 - 1}{t - 0} = \frac{t^2 - 46}{t - 14}.$$
By cross-multiplying and simplifying, we get the quadratic equation
$$14t^2 - 45t - 14 = 0$$
whose solutions are $t = -2/7$ and $7/2$. Therefore, the answer is $\boxed{7/2}$.

Problem 2 Let b and c be random integers from the set $\{1, 2, \ldots, 100\}$, chosen uniformly and independently. What is the probability that the roots of the quadratic $x^2 + bx + c$ are real? Express your answer as a fraction in simplest form.

Answer: $\dfrac{1743}{2000}$

Solution: The roots of a quadratic with integer coefficients are real when its discriminant is nonnegative. The discriminant of the quadratic x^2+bx+c is $b^2 - 4c$. We therefore need to determine the number of lattice points (b, c) with $1 \le b, c \le 100$ such that $b^2 - 4c \ge 0$, or $c \le b^2/4$.

If $b \ge 20$, then $b^2/4 \ge 100$, so all the lattice points (b, c) for which $b \ge 20$ and $1 \le c \le 100$ contribute, and these total 8100.

If $b = 1$, then $b^2/4 = 1/4$, so there is no lattice point of the form $(1, c)$ that contributes to our count.

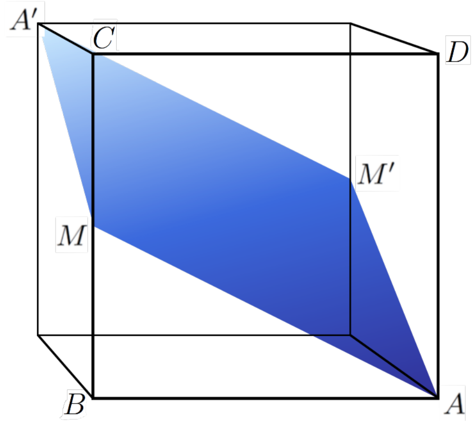
For $1 < b < 20$, note that the fractional part of $b^2/4$ is either 0 or $1/4$ depending on whether b is even or odd, respectively. Thus, the contribution from lattice points (b, c) with $1 < b < 20$ can be determined by tallying up the values $b^2/4$, but then adjusting for the fractional excesses by subtracting $1/4$ for each of the 9 odd numbers b in the range $1 < b < 20$. Using the algebraic identity for the sum of squares, namely $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$, we compute the contribution from lattice points (b, c) with $1 < b < 20$ to be

$$(\sum_{b=2}^{19} b^2/4) - 9(1/4) = \frac{1}{4}(\frac{19 \cdot 20 \cdot 39}{6} - 1) - 9/4 = 615.$$

Thus, the total is $8100 + 615 = 8715$ and since each lattice point is equally likely, the probability is $8715/10000 = \boxed{1743/2000}$.

Problem 3 Let $ABCD$ be a square face of a cube with edge length 2. A plane P that contains A and the midpoint of $\overline{BC}$ splits the cube into two pieces of the same volume. What is the square of the area of the intersection of P and the cube?

Answer: 24



Solution: Let M be the midpoint of $\overline{BC}$ and let A' and M' be the reflections of A and M through the center of the cube, respectively.

Because $\overline{AA'}$ and $\overline{MM'}$ intersect (at the center of the cube), they are contained in a common plane, and, by symmetry, this plane must split the cube in half, so must be the desired plane P. The intersection of P with the cube is the quadrilateral $AMA'M'$. Because each side of this quadrilateral extends across a square face of the cube from one vertex to the midpoint of a nonadjacent side, the quadrilateral is, in fact, a rhombus whose area is half the product of its diagonals. Since $AA' = \sqrt{12}$ and $MM' = \sqrt{8}$, the desired area is $\sqrt{12} \cdot \sqrt{8}/2 = \sqrt{24}$. Hence, the answer is $\boxed{24}$.

Problem 4 Determine the largest integer n such that $n < 103$ and $n^3 - 1$ is divisible by 103.

Answer: 56

Solution: We seek solutions to the equation

$$n^3 - 1 \equiv 0 \pmod{103}.$$

Now, $n^3 - 1 = (n-1)(n^2+n+1)$, so we seek to factor n^2+n+1, modulo 103. By the quadratic formula, the roots of n^2+n+1 are

$$\frac{-1 \pm \sqrt{-3}}{2}.$$

Since $-3 \equiv 100 \pmod{103}$, we conveniently find that 10 is a square root of -3 modulo 103. So these roots are $(-1 \pm 10)/2$ or 46 and 56, modulo 103. Thus, the desired answer is $\boxed{56}$.

Problem 5 Given a real number a, the *floor* of a, written $\lfloor a \rfloor$, is the greatest integer less than or equal to a. For how many real numbers x such that $1 \le x \le 20$ is

$$x^2 + \lfloor 2x \rfloor = \lfloor x^2 \rfloor + 2x\,?$$

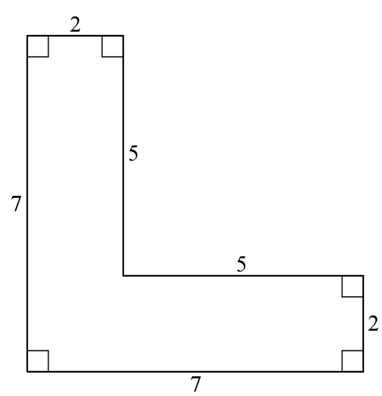
Answer: 362

Solution: The given equation can be rewritten as

$$x^2 - \lfloor x^2 \rfloor = 2x - \lfloor 2x \rfloor,$$

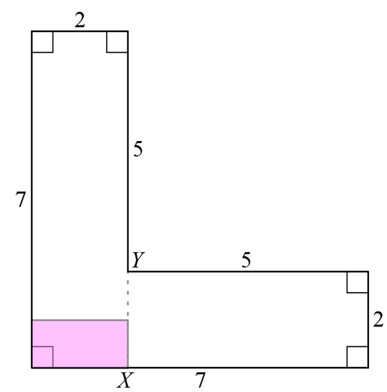
which says that the fractional parts of x^2 and $2x$ are equal, that is $x^2 - 2x$ is an integer. Now $x^2 - 2x$ is increasing and continuous for $1 \le x \le 20$, hence the number of solutions is equal to the number of integers between $1^2 - 2(1) = -1$ and $20^2 - 2(20) = 360$, inclusive. Therefore, the answer is $\boxed{362}$.

Problem 6 An L-shaped region is formed by attaching two 2 by 5 rectangles to adjacent sides of a 2 by 2 square as shown below.



The resulting shape has an area of 24 square units. How many ways are there to tile this shape with 2 by 1 dominos (each of which may be placed horizontally or vertically)?

Answer: 208



Solution: The bilateral symmetry of the shape induces an involution on the set of tilings. Because the lower left unit square in the figure must be covered either by a vertical or a horizontal domino, this involution has no fixed points, and the total number of tilings is twice the number of tilings where the lower left unit square is covered by a horizontal domino, which we now assume, as shown in the figure above.

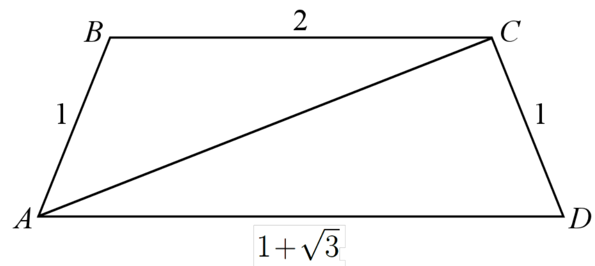
We claim that any such tiling can be split into tilings of two rectangles by cutting along the segment $\overline{XY}$. If a 2 by 1 domino (which has area 2) crosses over $\overline{XY}$, the untiled portions of each of the rectangles formed by cutting along $\overline{XY}$ will have area that is not equal to an even integer, making it impossible to tile them using 2 by 1 dominos. Therefore, the number of such tilings is equal to the product of the number of ways to tile a 2 by 6 rectangle and a 2 by 5 rectangle using 2 by 1 dominos.

The number of ways to tile a 2 by N rectangle, for positive integers N is equal to F_N, where F_N is the Fibonacci sequence indexed so that $F_0 = F_1 = 1$. (This can be shown by induction.)

Thus, the number of tilings where the lower left unit square is covered by a horizontal domino is equal to $F_5 F_6 = 8 \cdot 13 = 104$, and the total number of tilings of the given shape is $\boxed{208}$.

Problem 7 The quadrilateral $ABCD$ is an isosceles trapezoid with $AB = CD = 1$, $BC = 2$, and $DA = 1 + \sqrt{3}$. What is the measure of $\angle ACD$ in degrees?

Answer: 90



Solution: Applying the law of cosines to each of the triangles shown in the figure, we find that

$$(AC)^2 = 1^2 + 2^2 - 2(1)(2)\cos B$$

$$(AC)^2 = 1^2 + (1+\sqrt{3})^2 - 2(1)(1+\sqrt{3})\cos D$$

Because angles B and D are supplementary, $\cos B = -\cos D$. If we add twice the second equation to $1+\sqrt{3}$ times the first, we get

$$(1+\sqrt{3}+2)(AC)^2 = (1+\sqrt{3})(1^2+2^2) + 2(1^2+(1+\sqrt{3})^2).$$

This simplifies to $(AC)^2 = \frac{15+9\sqrt{3}}{3+\sqrt{3}} = 3+2\sqrt{3} = (1+\sqrt{3})^2 - 1^2$. By the converse of the Pythagorean theorem, $\angle ACD$ is $\boxed{90}$ degrees.

Problem 8 Let S be the set of numbers of the form $n^5 - 5n^3 + 4n$, where n is an integer that is not a multiple of 3. What is the largest integer that is a divisor of every number in S?

Answer: 360

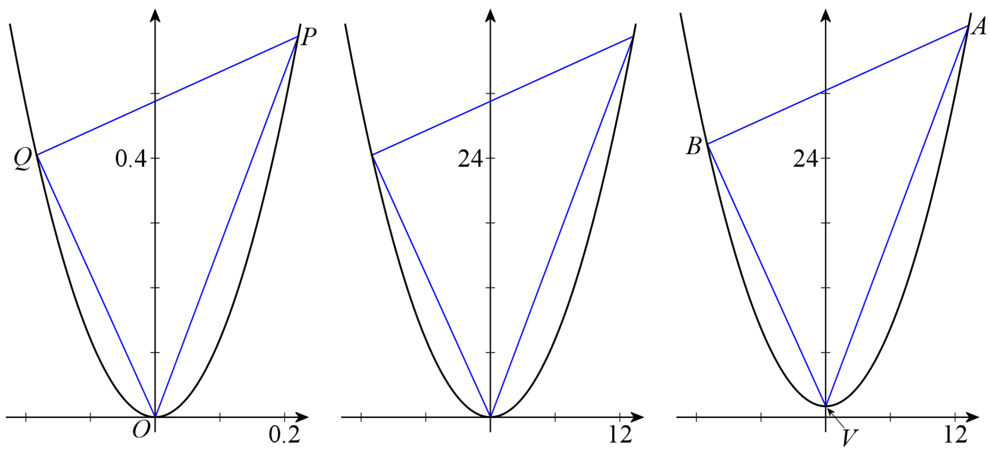
Solution: Let $p(n) = n^5 - 5n^3 + 4n$. Note that

$$p(n) = n(n^2-1)(n^2-4) = (n-2)(n-1)n(n+1)(n+2),$$

which is a product of 5 consecutive integers, where the middle integer is not a multiple of 3. In such a string of 5 consecutive integers, at least one must be divisible by 5. at least two must be divisible by 3, and at least one must be divisible by 4, and the number 2 above or 2 below the multiple of 4 must also be among the string of integers. Thus, any such number is divisible by at least $2^3 3^2 5 = 360$. On the other hand, the greatest common divisor of $p(4) = 360 \cdot 2$ and $p(5) = 360 \cdot 7$ is 360. Hence, the answer is $\boxed{360}$.

Problem 9 Let $\triangle PQO$ be the unique right isosceles triangle inscribed in the parabola $y = 12x^2$ with P in the first quadrant, right angle at Q in the second quadrant, and O at the vertex $(0, 0)$. Let $\triangle ABV$ be the unique right isosceles triangle inscribed in the parabola $y = x^2/5 + 1$ with A in the first quadrant, right angle at B in the second quadrant, and V at the vertex $(0, 1)$. The y-coordinate of A can be uniquely written as $uq^2 + vq + w$, where q is the x-coordinate of Q and u, v, and w are integers. Determine $u + v + w$.

Answer: 661



Left: $y = 12x^2$. Center: $y = x^2/5$. Right: $y = x^2/5 + 1$.

Solution: We are given that $Q = (q, 12q^2)$. Since $\angle PQO$ is a right angle and $QO = QP$, we find that $P = (q + 12q^2, 12q^2 - q)$.

We use the fact that all parabolas are similar to each other and all right isosceles triangles are similar to each other. The parabola $y = x^2/5 + 1$ can be obtained from the parabola $y = 12x^2$ by scaling by a factor of 60 and then translating up 1 unit. Thus, $A = 60(q + 12q^2, 12q^2 - q) + (0, 1)$. Hence, the y-coordinate of A is $720q^2 - 60q + 1$ and the answer is $720 - 60 + 1 = \boxed{661}$.

Problem 10 An algal cell population is found to have a_k cells on day k. Each day, the number of cells at least doubles. If $a_0 \geq 1$ and $a_3 \leq 60$, how many quadruples of integers (a_0, a_1, a_2, a_3) could represent the algal cell population size on the first 4 days?

Answer: 8344

Solution: We are given that $a_3 \geq 2a_2 \geq 2a_1 \geq 2a_0$. Thus, $a_3 \geq 8a_0$, and since $a_3 \leq 60$, we find that $1 \leq a_0 \leq 7$. Similarly, we find that $a_1 \leq 15$ and $a_2 \leq 30$. The number of possibilities is therefore given by

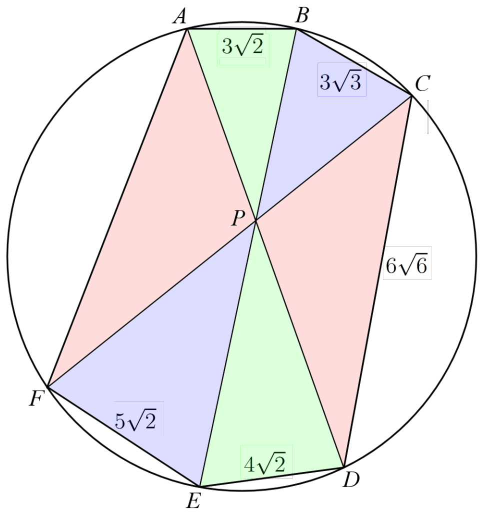
$$\sum_{a_0=1}^{7} \sum_{a_1=2a_0}^{15} \sum_{a_2=2a_1}^{30} \sum_{a_3=2a_2}^{60} 1.$$

One way to compute this is as follows:

$$\begin{aligned}
\sum_{a_0=1}^{7} \sum_{a_1=2a_0}^{15} \sum_{a_2=2a_1}^{30} \sum_{a_3=2a_2}^{60} 1 &= \sum_{a_0=1}^{7} \sum_{a_1=2a_0}^{15} \sum_{a_2=2a_1}^{30} (61 - 2a_2) \\
&= \sum_{a_0=1}^{7} \sum_{a_1=2a_0}^{15} \sum_{k=1}^{31-2a_1} (2k - 1) \\
&= \sum_{a_0=1}^{7} \sum_{a_1=2a_0}^{15} (31 - 2a_1)^2 \\
&= \sum_{a_0=1}^{7} \sum_{k=1}^{16-2a_0} (2k - 1)^2 \\
&= 7(1^2 + 3^2) + 6(5^2 + 7^2) + 5(9^2 + 11^2) + 4(13^2 + 15^2) \\
&\quad + 3(17^2 + 19^2) + 2(21^2 + 23^2) + (25^2 + 27^2) \\
&= \boxed{8344}.
\end{aligned}$$

Problem 11 Let A, B, C, D, E, and F be 6 points around a circle, listed in clockwise order. We have $AB = 3\sqrt{2}$, $BC = 3\sqrt{3}$, $CD = 6\sqrt{6}$, $DE = 4\sqrt{2}$, and $EF = 5\sqrt{2}$. Given that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent, determine the square of AF.

Answer: 225



Solution: Let P be the point of concurrency of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Because inscribed angles that subtend equal arcs are congruent, we see that $\triangle PAB \cong \triangle PED$, $\triangle PBC \cong \triangle PFE$, and $\triangle PCD \cong \triangle PAF$. (In the figure above, triangles of the same color are similar.) Therefore,

$$\frac{AB}{DE} \cdot \frac{CD}{FA} \cdot \frac{EF}{BC} = \frac{PA}{PE} \cdot \frac{PC}{PA} \cdot \frac{PE}{PC} = 1.$$

Hence,

$$\frac{3\sqrt{2}}{4\sqrt{2}} \cdot \frac{6\sqrt{6}}{FA} \cdot \frac{5\sqrt{2}}{3\sqrt{3}} = 1,$$

from which it follows that $FA = 15$ and the answer is $\boxed{225}$.

Problem 12 Solve the equation

$$\sin 9^\circ \sin 21^\circ \sin(102^\circ + x^\circ) = \sin 30^\circ \sin 42^\circ \sin x^\circ$$

for x where $0 < x < 90$.

Answer: 9

Solution: Using the double-angle formula for sine and the fact that $\sin 30^\circ = 1/2$, we see that the equation is equivalent to

$$\sin 9^\circ \sin(102^\circ + x^\circ) = \cos 21^\circ \sin x^\circ,$$

which can be rewritten

$$\sin 9^\circ \cos(12^\circ + x^\circ) = \cos 21^\circ \sin x^\circ,$$

since $\sin(90^\circ + z) = \cos z$. By comparing the arguments of corresponding trigonometric functions, we can see that $x = 9$ is a solution. However, without this observation, we can use the identity

$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B)) \tag{1}$$

to rewrite this equation as

$$\sin(21^\circ + x^\circ) + \sin(-3^\circ - x^\circ) = \sin(x^\circ + 21^\circ) + \sin(x^\circ - 21^\circ),$$

which is equivalent to

$$\sin(x^\circ + 3^\circ) + \sin(x^\circ - 21^\circ) = 0.$$

Using the identity (1) again, this is equivalent to

$$2\sin(x^\circ - 9^\circ)\cos(12^\circ) = 0,$$

from which we deduce that $\sin(x^\circ - 9^\circ) = 0$. For $x \in [0, 90]$, this last equation has the unique solution $x = \boxed{9}$.

Problem 13 The roots of the polynomial $x^4 - 4ix^3 + 3x^2 - 14ix - 44$ form the vertices of a parallelogram in the complex plane. What is the area of the parallelogram?

Answer: 12

Solution: Since the roots form the vertices of a parallelogram, the intersection of the parallelogram's diagonals must be located at the average of the sum of the roots, which, by Vieta's formulas, is located at i. If we translate the polynomial by $-i$, the resulting roots will be invariant under the transformation $z \to -z$, and so will be a polynomial in x^2. Thus,

$$(x+i)^4 - 4i(x+i)^3 + 3(x+i)^2 - 14i(x+i) - 44 = x^4 + 9x^2 - 36.$$

Now $x^4 + 9x^2 - 36 = (x^2 + 12)(x^2 - 3)$, whose roots are $\pm\sqrt{12}$ and $\pm\sqrt{-3}$. Thus, the parallelogram is, in fact, a rhombus, whose diagonal lengths are $2\sqrt{12}$ and $2\sqrt{3}$, which has area $\frac{1}{2}(2\sqrt{12})(2\sqrt{3}) = \boxed{12}$.

Problem 14 Across the face of a rectangular post-it note, you idly draw lines that are parallel to its edges. Each time you draw a line, there is a 50% chance it'll be in each direction and you never draw over an existing line or the edge of the post-it note. After a few minutes, you notice that you've drawn 20 lines. What is the expected number of rectangles that the post-it note will be partitioned into?

Answer: 116

Solution: The expected number of rectangles added to the initial rectangle is the sum of the expected number of new rectangles obtained with each new line that is drawn for each of 20 new lines. When the kth line is drawn, there are already $k-1$ lines across the post-it note. Suppose that v of these $k-1$ lines are vertical and h are horizontal, so that $v+h=k-1$. Then there is a 50% chance that $v+1$ new rectangles are formed and a 50% chance that $h+1$ new rectangles are formed, depending on whether the new line is horizontal or vertical, respectively. The expected number of new rectangles obtained when the kth line is drawn is therefore

$$\frac{1}{2}((v+1)+(h+1)) = \frac{k+1}{2}.$$

Thus, the expected number of rectangles after 20 lines are drawn is

$$1 + \sum_{k=1}^{20} \frac{k+1}{2} = \boxed{116}.$$

Problem 15 What is the smallest positive integer m such that $15!\,m$ can be expressed in more than one way as a product of 16 distinct positive integers, up to order?

Answer: 24

Solution: We claim that $m = 24$.

First, note that $15! \cdot 24 = 14! \cdot 18 \cdot 20 = 11! \cdot 13 \cdot 14 \cdot 15 \cdot 16 \cdot 18$.

Now suppose that $m < 24$ and suppose that $15! \cdot m$ can be written in another way as a product $p_1 p_2 p_3 \cdots p_{16}$, where the p_k are positive integers with $p_k < p_j$ whenever $k < j$. (So $p_k \neq k$ for $k \leq 15$ and $p_{16} \neq m$.)

The smallest possible product of 16 distinct positive integers is 16!, hence $m > 16$.

Suppose $p_k > k$ for some k with $1 \le k \le 16$. Then $p_1 p_2 p_3 \cdots p_{16} \ge 17!/k$. However $17!/k > 15! \cdot m$ for $k < 17 \cdot 16/24 = 11\frac{1}{3}$, hence $p_k = k$ for all $k \le 11$. We therefore have an equation of the form

$$12 \cdot 13 \cdot 14 \cdot 15 \cdot m = p_{12} p_{13} p_{14} p_{15} p_{16}. \tag{2}$$

where $p_{12} \ge 12$. Because $12 \cdot 13 \cdot 14 \cdot 15$ is the smallest possible product of 4 distinct positive integers all greater than 11, we must have $p_{16} < m$, for otherwise, $p_k = k$ for all $k \le 15$ and $p_{16} = m$, a contradiction. Also, m cannot be 17, 19, 22, or 23, as the largest prime factor of the left-hand side of (2) could not occur as a factor of the right-hand side of (2). So we've reduced the problem to checking the possibilities: $m = 18$, 20, or 21.

If $m = 18$, then $16 \le p_{16} \le 17$. But $p_{16} = 16$ is not possible because then the right-hand side of (2) would be the smallest possible product (of 5 distinct integers all greater than 11), and $p_{16} = 17$ is not possible because the left-hand side of (2) has no prime factor of 17.

If $m = 20$, the left-hand side of (2) would be divisible by 5^2. This forces $p_{16} = 20$, a contradiction.

Similarly, if $m = 21$, the left-hand side of (2) is divisible by 7^2, forcing $p_{16} = 21$, a contradiction.

Therefore, the answer is, indeed, $\boxed{24}$.

Problem 16 A snail begins a journey starting at the origin of a coordinate plane. The snail moves along line segments of length $\sqrt{10}$ and in any direction such that the horizontal and vertical displacements are both integers. As the snail moves, it leaves a trail tracing out its entire journey. After a while, this trail can form various polygons. What is the smallest possible area of a polygon that could be created by the snail's trail? Express your answer as a fraction in simplest form.

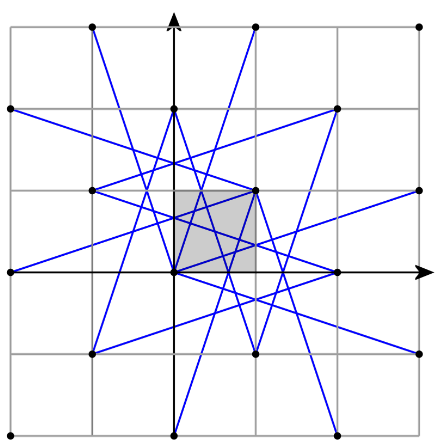
Answer: $\dfrac{1}{60}$

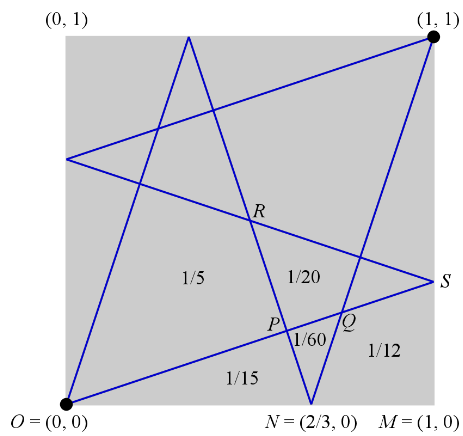
Solution: The allowable vector displacements for the snail are $(\pm 3, \pm 1)$ and $(\pm 1, \pm 3)$. Because the sum of these displacements is even, the parity of the sum of the coordinates of the lattice points that the snail can visit is invariant. Since the snail begins at the origin, the snail can only visit lattice points (x, y) for which $x + y$ is even. Conversely, the vector equation

$$(x, y) = \frac{x+y}{2}((1,3) + (1,-3)) + y((-3,1) + (1,-3) + (1,3))$$

shows that the snail can reach any lattice point (x, y) such that $x+y$ is even.

We conclude that every leg of the snail's journey consists of an allowable vector displacement between lattice points whose coordinate sums are even. Now imagine that the snail traces out *every* such line segment. Because the set of possible displacements enjoys the same symmetries as that of a square and all unit squares with lattice vertices have precisely two diagonally opposite vertices whose coordinate sums are even, every such unit square will exhibit the same pattern of snail legs, up to symmetry. Therefore, the smallest possible polygon can be found be looking for the smallest possible polygon that intersects the interior of any such unit square. (Even though we deduce this by imagining that the snail traces out an infinite path, the polygons that intersect with a given unit square can all be formed by a finite path because there will only be finitely many legs involved with those polygons.)





The left figure above highlights the unit square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, together with every snail leg that bounds a polygonal region which intersects its interior. The right figure is an enlargement of the highlighted unit square and gives the area of a polygonal region from each congruence class.

Using linear algebra, we compute that $N = (2/3, 0)$, $P = (3/5, 1/5)$, $Q = (3/4, 1/4)$, $S = (1, 1/3)$, and $R = (1/2, 1/2)$. E.g., P is the intersection of the lines $y = \frac{1}{3}x$ and $y = 2 - 3x$. We then compute the areas of the regions that the unit square is split into by the legs of the snail's journey using

Gauss's shoelace formula or some other method adapted to the situation. E.g., the area of PQN is the area of OMS less the areas of ONP and $NQSM$, and the area of $NQSM$ is twice the area (by symmetry) of NQM.

Since the polygons that are not contained within the unit square have more area than the area of their intersection with the unit square, we see that the smallest possible area of a polygon bounded by the legs of the snail's trail is $\boxed{1/60}$.

Note: Some contestants misinterpreted the problem by considering only those polygons whose sides consist of the displacements $(\pm 3, \pm 1)$ and $(\pm 1, \pm 3)$. Since the problem does not qualify the term "polygon," we must consider any and all polygons created by a snail trail. If the problem were about this restricted class of polygons, the problem statement would require some kind of qualification, such as by restricting to non-intersecting trails.

Problem 17 Let O be the set of odd numbers between 0 and 100. Let T be the set of subsets of O of size 25. For any finite subset of integers S, let $P(S)$ be the product of the elements of S. Define $n = \sum_{S \in T} P(S)$. If you divide n by 17, what is the remainder? (The remainder is an integer between 0 and 16, inclusive.)

Answer: 15

Solution: By Vieta's formulas, m is the coefficient of x^{25} in

$$p(x) = (x–1)(x–3)(x–5) \cdots (x–99).$$

By Fermat's little theorem, we have

$$p(x) = \frac{(x^{17}–x)^3}{x+1} = \frac{x^{51}–3x^{35} + 3x^{19}–x^3}{x+1} \pmod{17}.$$

Using polynomial long division, we find

$$p(x) = (x-1)(\sum_{k=2}^{9} x^{2k-1} - 2\sum_{k=10}^{17} x^{2k-1} + \sum_{k=18}^{25} x^{2k-1}) \pmod{17},$$

from which we can see that the coefficient of x^{25} is -2, or $\boxed{15}$ modulo 17.

Problem 18 Let A be the locus of points (α, β, γ) in the $\alpha\beta\gamma$-coordinate space that satisfy the following properties:

I We have α, β, $\gamma > 0$.

II We have $\alpha + \beta + \gamma = \pi$.

III The intersection of the three cylinders in the xyz-coordinate space given by the equations

$$\begin{aligned} y^2 + z^2 &= \sin^2 \alpha \\ z^2 + x^2 &= \sin^2 \beta \\ x^2 + y^2 &= \sin^2 \gamma \end{aligned}$$

is nonempty.

Determine the area of A. Express your answer in terms of π.

Answer: $\dfrac{\sqrt{3}}{8}\pi^2$

Solution: By conditions I and II and the law of sines, $\sin\alpha$, $\sin\beta$, and $\sin\gamma$ are the side lengths of a triangle with angles α, β, and γ (and a circumcircle of diameter 1).

If we subtract each equation in condition III from the sum of the other two, we find

$$\begin{aligned} x^2 &= \frac{1}{2}(\sin^2\beta + \sin^2\gamma - \sin^2\alpha) \\ y^2 &= \frac{1}{2}(\sin^2\gamma + \sin^2\alpha - \sin^2\beta) \\ z^2 &= \frac{1}{2}(\sin^2\alpha + \sin^2\beta - \sin^2\gamma) \end{aligned}$$

Thus, the cylinders intersect if and only if $\sin^2\beta + \sin^2\gamma - \sin^2\alpha \geq 0$, $\sin^2\gamma + \sin^2\alpha - \sin^2\beta \geq 0$, and $\sin^2\alpha + \sin^2\beta - \sin^2\gamma \geq 0$. In other words, if and only if the triangle with sides $\sin\alpha$, $\sin\beta$, and $\sin\gamma$ is not an obtuse triangle, which is equivalent to $\alpha \leq \pi/2$, $\beta \leq \pi/2$, and $\gamma \leq \pi/2$. The set of (α, β, γ) that satisfy the first two conditions of the problem corresponds to the equilateral triangle with vertices $(\pi, 0, 0)$, $(0, \pi, 0)$, and $(0, 0, \pi)$ in the $\alpha\beta\gamma$-coordinate space. Of these (α, β, γ), those that also satisfy $\leq \pi/2$, $\beta \leq \pi/2$, and $\gamma \leq \pi/2$ constitute the medial triangle whose vertices are $(\pi/2, \pi/2, 0)$, $(\pi/2, 0, \pi/2)$, and $(0, \pi/2, \pi/2)$. This is an equilateral triangle with side length $\frac{\sqrt{2}}{2}\pi$ and area $\boxed{\dfrac{\sqrt{3}}{8}\pi^2}$.

Problem 19 Let S_- be the semicircular arc defined by

$$(x+1)^2 + (y - \frac{3}{2})^2 = \frac{1}{4} \text{ and } x \leq -1.$$

Let S_+ be the semicircular arc defined by

$$(x-1)^2 + (y - \frac{3}{2})^2 = \frac{1}{4} \text{ and } x \geq 1.$$

Let R be the locus of points P such that P is the intersection of two lines, one of the form $Ax + By = 1$ where $(A, B) \in S_-$ and the other of the form $Cx + Dy = 1$ where $(C, D) \in S_+$. What is the area of R? Express your answer as a fraction in simplest form.

Answer: $\frac{1}{6}$

Solution: Let $(A, B) \in S_-$ and $(C, D) \in S_+$. Suppose that (X, Y) is the point of intersection of the lines $Ax + By = 1$ and $Cx + Dy = 1$. Then we have both $AX + BY = 1$ and $CX + DY = 1$. In other words, (A, B) and (C, D) are both on the line $xX + yY = 1$, and, in fact, all points (P, Q) on the line $xX + yY = 1$ correspond to lines $Px + Qy = 1$ that contain (X, Y). Therefore, the point of intersection (X, Y) is also the point of intersection of any two lines $A'x + B'y = 1$ and $C'x + D'y = 1$ where (A', B') and (C', D') are distinct points on the line $xX + yY = 1$. (Note that lines of the form $Ax + By = 1$ never contain the origin, so $(X, Y) \neq (0, 0)$.) Because a line passes through S_- and S_+ if and only if it also passes through the vertical line segments $\{-1\} \times [1, 2]$ and $\{1\} \times [1, 2]$, we can simplify the algebra by assuming instead that $(A, B) \in \{-1\} \times [1, 2]$ and $(C, D) \in \{1\} \times [1, 2]$.

Thus, the problem is equivalent to finding the area of the locus of intersections of lines of the from $-x + By = 1$ and $x + Dy = 1$, where $B, D \in [1, 2]$. The intersection of these two lines is the point $(\frac{B-D}{B+D}, \frac{2}{B+D})$.

We make the following change of coordinates: $z = B - D$, $w = B + D$. The region $[1, 2] \times [1, 2]$ in the BD-plane transforms to the square with vertices $(0, 2)$, $(1, 3)$, $(0, 4)$, and $(-1, 3)$ in the zw-plane. Let this square be S. The region R is the set of points $(z/w, 2/w)$ for $(z, w) \in S$. Observe that under the transformation $(z, w) \mapsto (z/w, 2/w)$, horizontal line segments map to horizontal line segments, and the ordering of points on a horizontal line above the horizontal axis is preserved. Therefore, the boundary of R corresponds to the image of the boundary of S. The boundary of S consists of the line segments

$$\begin{aligned} w &= 2 + z, \quad z \in [0, 1] \\ w &= 4 - z, \quad z \in [0, 1] \\ w &= 2 - z, \quad z \in [-1, 0] \\ w &= 4 + z, \quad z \in [-1, 0] \end{aligned}$$

Substituting these into the formulas for the transformation, we find 4 line segments that form the boundary of a kite with vertices $(0, 1/2)$, $(1/3, 2/3)$, $(0, 1)$, and $(-1/3, 2/3)$. For example, when $w = 2 + z$, $z \in [0, 1]$, we get the points $(1 - 2/w, 2/w)$, which form a line segment connecting $(0, 1)$ to $(1/3, 2/3)$. The area of this kite is $\frac{1}{2}(1/2)(2/3) = \boxed{1/6}$.

Problem 20 Let $a_n = n(2n + 1)$. Evaluate

$$\left| \sum_{1 \le j < k \le 36} \sin\left(\frac{\pi}{6}(a_k - a_j)\right) \right|.$$

Answer: 18

> "*This, I don't understand. That's why I call it* 'algebra.' *But this, I understand. That's why I call it* 'geometry.'" - Gerald Sacks (during a lecture at Harvard while pointing at different parts of a proof).

Solution: Let v_k be the vector $(\cos(\frac{\pi a_k}{6}), \sin(\frac{\pi a_k}{6}))$. Then

$$\sin\left(\frac{\pi}{6}(a_k - a_j)\right) = \sin\left(\frac{\pi a_k}{6}\right) \cos\frac{\pi a_j}{6}\Big) - \cos\left(\frac{\pi a_k}{6}\right) \sin\left(\frac{\pi a_j}{6}\right) = \det(v_j, v_k).$$

Thus,

$$\begin{aligned} \left| \sum_{1 \le j < k \le 36} \sin\left(\frac{\pi}{6}(a_k - a_j)\right) \right| &= \left| \sum_{k=1}^{35} \det\left(\sum_{j=1}^{k} v_j, v_{k+1}\right) \right| \\ &= |\det(v_1, v_2) \\ &\quad + \det(v_1 + v_2, v_3) \\ &\quad + \det(v_1 + v_2 + v_3 + v_4, v_5) \\ &\quad + \dots \\ &\quad + \det(v_1 + v_2 + v_3 + \dots + v_{34}, v_{35}) \\ &\quad + \det(v_1 + v_2 + v_3 + v_4 + \dots + v_{35}, v_{36})|. \end{aligned}$$

This determinant sum expresses twice the area enclosed by the path (see figure below) obtained by starting at the origin and taking steps v_1, then v_2, then v_3, then v_4, etc., multiplied by the number of times the boundary is lapped. One way to see this is by interpreting the determinant as the signed area of a parallelogram and another way is to recognize that the sum is what is prescribed by Gauss' shoelace formula, up to a factor of 2.

Note that

$$\begin{aligned} a_{n+3} &= (n+3)(2(n+3)+1) \\ &= a_n + 12n + 21 \\ &= a_n - 3 \pmod{12}. \end{aligned}$$

Therefore, the path can be formed by taking the first 3 steps as a "path snippet" and extending the path repeatedly by successive 90° clockwise rotations of the path snippet. Since $a_1 = 3$, $a_2 = 10$, and $a_3 = 21$, the path snippet comprises 3 sides of a rhombus with unit side lengths and interior angles of 30° and 150°. The resulting path is therefore the dodecagon formed by attaching 4 such parallelograms onto the 4 sides of a unit square as shown in the figure below. This description also shows that the area of the dodecagon is 3 (the area of the central unit square plus 4 rhombi each of area 1/2) and the limits of summation correspond to the path looping around the boundary 3 times. Hence the answer is $3 \cdot 2 \cdot 3 = \boxed{18}$.