Math Prize for Girls Olympiad

Advantage Testing Foundation/Jane Street

Time Limit: 4 hours

Thursday, November 30, 2023

1. Let $n \geq 2023$ be an integer. Prove that there exists a permutation $(p_1, p_2, \dots, p_n)$ of $(1, 2, \dots, n)$ such that

$$p_1 + 2p_2 + 3p_3 + \cdots + np_n$$

is divisible by n.

2. The two cats Fitz and Will play the following game. On a blackboard is written the expression

$$x^{100} + \Box x^{99} + \Box x^{98} + \Box x^{97} + \cdots + \Box x^2 + \Box x + 1.$$

Both cats take alternate turns replacing one $\Box$ with a 0 or 1, with Fitz going first, until (after 99 turns) all the blanks have been filled. If the resulting polynomial obtained has a real root, then Will wins, otherwise Fitz wins. Determine, with proof, which player has a winning strategy.

3. Let m be the product of the first 100 primes, and let S denote the set of divisors of m greater than 1 (hence S has exactly $2^{100} - 1$ elements). We wish to color each element of S with one of k colors such that

   - every color is used at least once; and
   - any three elements of S whose product is a perfect square have exactly two different colors used among them.

Find, with proof, all values of k for which this coloring is possible.

4. Let $O = (0, 0)$ be the origin of the xy-plane. We say a lattice triangle ABC is *marine* if it has centroid O and area $\frac{3}{2}$.

Let P be any point in the plane which is not a lattice point. Prove that P lies in the interior of some marine triangle if and only if the line segment $\overline{OP}$ does not pass through any lattice points besides O.

(A *lattice point* is a point whose x-coordinate and y-coordinate are both integers. A *lattice triangle* is a triangle whose vertices are lattice points.)