The Fifteenth Annual Math Prize for Girls

Sunday, October 8, 2023

Version A

DIRECTIONS

1. Do not open this test until your proctor instructs you to.
2. Fill out the top of your answer sheet. Your student number is on your name tag. Your test version (A, B, C, or D) is above on this page.
3. You may use pencils, pens, and erasers. You may also use the scratch paper that we provide. You may NOT use calculators, rulers, protractors, compasses, graph paper, or books.
4. Write your final answers on the answer sheet. When a problem specifies a particular form for the answer, write your answer in that form. Write your answers clearly. We will collect only the answer sheet from you.

NOTES

1. This test contains 20 problems. You will have 150 minutes (2.5 hours) to take the test. Your score will be the number of correct answers.
2. Figures are not necessarily drawn to scale.
3. Good luck!

1. The frame of a painting has the form of a $105''$ by $105''$ square with a $95''$ by $95''$ square removed from its center. The frame is built out of congruent isosceles trapezoids with angles measuring $45°$ and $135°$. Each trapezoid has one base on the frame's outer edge and one base on the frame's inner edge. Each outer edge of the frame contains an odd number of trapezoid bases that alternate long, short, long, short, etc. What is the maximum possible number of trapezoids in the frame?

2. In the xy-coordinate plane, the horizontal line $y = k$ intersects the graph of the cubic $2x^3 + 6x^2 - 4x + 5$ in three points P, Q, and R. Given that Q is the midpoint of P and R, what is k?

3. You have 5000 distinct finite sets. Their intersection is empty. However, the intersection of any two is nonempty. What is the smallest possible number of elements contained in their union?

4. Let $\triangle A_1A_2A_3$ be an equilateral triangle with unit side length. For $k = 1$, 2, and 3, let B_k be the point on the boundary of $\triangle A_1A_2A_3$ located $1/3$ unit away from A_k in the clockwise direction and let C_k be the point on the boundary of $\triangle A_1A_2A_3$ located $1/3$ unit away from A_k in the counterclockwise direction. What fraction of the area of $\triangle A_1A_2A_3$ is the area of the intersection of $\triangle B_1B_2B_3$ and $\triangle C_1C_2C_3$? Express your answer as a fraction in simplest form.

5. Acute triangle ABC has area 870. The triangle whose vertices are the feet of the altitudes of $\triangle ABC$ has area 48. Determine

$$\sin^2 A + \sin^2 B + \sin^2 C.$$

Express your answer as a fraction in simplest form.

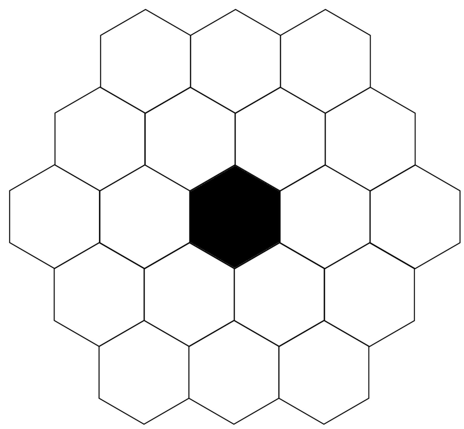
6. Solve for x:

$$\begin{aligned} v - w + x - y + z &= 79 \\ v + w + x + y + z &= -1 \\ v + 2w + 4x + 8y + 16z &= -2 \\ v + 3w + 9x + 27y + 81z &= -1 \\ v + 5w + 25x + 125y + 625z &= 79. \end{aligned}$$

7. An arithmetic expression is created by inserting either a plus sign or a multiplication sign in each of the 11 spaces between consecutive $\sqrt{3}$'s in a row of twelve $\sqrt{3}$'s. The signs are chosen uniformly and independently at random. What is the probability that the resulting expression evaluates to $12\sqrt{3}$? Express your answer as a fraction in simplest form.

8. For a positive integer n, let $p(n)$ denote the number of distinct prime numbers that divide evenly into n. Determine the number of solutions, in positive integers n, to the inequality $\log_4 n \le p(n)$.

9. The ring shown below is made out of 18 congruent regular hexagons. How many ways are there to tile the ring using tiles that consist of two hexagons, each congruent to any one of the 18 in the design, joined edge-to-edge? (The central hexagon, in black, is not to be covered with a tile and the ring cannot be rotated or reflected.)



10. Find all integers x between 0 and the prime number 4099 such that $x^3 - 3$ is divisible by 4099. Write your answer as a list of integers (in any order).

11. A random triangle is produced as follows. A pair of standard dice is rolled independently three times to get three random numbers between 2 and 12, inclusive, by adding the numbers that come up on each pair rolled. Call these three random numbers a, b, and t. The random triangle has two sides of lengths a and b with the angle between them measuring $15(t-1)$ degrees. What is the probability that the triangle is a right triangle? Express your answer as a fraction in simplest form.

12. Let S be the set of fractions of the form $\frac{\operatorname{lcm}(A,B)}{A+B}$, where A and B are positive integers and $\operatorname{lcm}(A,B)$ is the least common multiple of A and B. What is the smallest number exceeding 3 in S? Express your answer as a fraction in simplest form.

13. Let

$$f(t) = \frac{(10+9i)t - 10 + 9i}{t+i},$$

where $i = \sqrt{-1}$. Let $P = f(0)$, $Q = f(2023)$, and $R = f(1)$. Determine $\sin^2(m\angle PQR)$. Express your answer as a fraction in simplest form.

14. Five points are chosen uniformly and independently at random on the surface of a sphere. Next, 2 of these 5 points are randomly picked, with every pair equally likely. What is the probability that the 2 points are separated by the plane containing the other 3 points? Express your answer as a fraction in simplest form.

15. A square is divided into four non-overlapping isosceles triangles. Let X be the degree measure of one of the twelve angles of these four triangles. Compute the sum of all possible different values of X. (Consider all possible diagrams.)

16. Let $f(x) = x^2 - 3/4$. Let $f^{(n)}(x)$ denote the composition of f with itself n times. For example, $f^{(3)}(x) = f(f(f(x)))$. Let R be the set of complex numbers that is the union of the roots of the polynomials $f^{(n)}(x^2 + 3/4)$ over positive integers n. Let B be the smallest rectangle in the complex plane with sides parallel to the real and imaginary axes that contains R. What is the square of the area of B?

17. Let C be a unit cube. Let D be a translate of C such that one corner of D is located at the center of C and one corner of C is located at the center of D. Let D' be the image of D under a 60° clockwise rotation about the line that passes through both cube centers when looking from the center of D to the center of C. What is the volume of the intersection of C with D'? Express your answer as a fraction in simplest form.

18. A unit square is decorated with snippets of the graph of $y = x^2$ as follows: We consider the graph of $y = x^2$ restricted to the domain $0 \le x \le 6$. We cut up the first quadrant (the positive quadrant) into unit squares with lattice vertices. We translate each square so that they are stacked, one on top of the other. We merge all of these squares. How many regions is the unit square divided into by all the overlaid snippets of the graph of the parabola?

19. Let $N = \displaystyle\prod_{k=1}^{1000} (4^k - 1)$. Determine the largest positive integer n such that 5^n divides evenly into N.

20. Let $f_1(x) = 2\pi \sin(x)$. For $n > 1$, define $f_n(x)$ recursively by

$$f_n(x) = 2\pi \sin(f_{n-1}(x)).$$

How many intervals $[a, b]$ are there such that

- $0 \le a < b \le 2\pi$,
- $f_6(a) = -2\pi$,
- $f_6(b) = 2\pi$,
- and f_6 is increasing on $[a, b]$?