Math Prize for Girls
2023 Solutions

Problem 1 The frame of a painting has the form of a $105''$ by $105''$ square with a $95''$ by $95''$ square removed from its center. The frame is built out of congruent isosceles trapezoids with angles measuring $45°$ and $135°$. Each trapezoid has one base on the frame's outer edge and one base on the frame's inner edge. Each outer edge of the frame contains an odd number of trapezoid bases that alternate long, short, long, short, etc. What is the maximum possible number of trapezoids in the frame?

Answer: 76

Solution: Let L be the length of the long base of the trapezoid and let S be the length of the short base (both in inches). Since the frame is $5''$ thick, the height of the trapezoid is $5''$. Because the trapezoid has angles of $45°$ and $135°$, we have $L = S + 10$. Along one side of the frame, if there are n long bases, then there are $n - 1$ short bases. The total length of these bases is $Ln + S(n - 1) = n(2S + 10) - S = 105$. If S is fixed, the expression $n(2S + 10) - S$ increases with n, and therefore, if we increase n, we must decrease S to maintain equality in this last equation. When $S = 0$, we find $n = 105/10$, which is between 10 and 11. Therefore, the largest n can be is 10, which corresponds to 19 trapezoids on each side for a total of $\boxed{76}$ trapezoids in the frame.

Alternatively, imagine the limiting case where the trapezoid becomes a right triangle with hypotenuse $10''$. In this case, at most 10 such triangle bases can be fit into a side, for a total of 19 triangles per side. We can then imagine expanding these triangles into trapezoids until we get a snug fit.

Problem 2 In the xy-coordinate plane, the horizontal line $y = k$ intersects the graph of the cubic $2x^3 + 6x^2 - 4x + 5$ in three points P, Q, and R. Given that Q is the midpoint of P and R, what is k?

Answer: 13

Solution: If the graph is translated so that Q becomes the origin of coordinates, then the cubic has the form $2x(x + r)(x - r)$, where $r = QR$. The cubic $2x(x+r)(x-r)$ has no quadratic term, and vertical translation does not change this property. By Vieta's formulas, the horizontal translation which results in loss of the quadratic term is a horizontal shift by $6/(2(3)) = 1$ unit to the right. Note that k is the y-coordinate of any of the points P, Q, or R, or their images under horizontal translation. Hence, k is the value of the cubic at $x = -1$, which is $-2+6+4+5 = \boxed{13}$.

Problem 3 You have 5000 distinct finite sets. Their intersection is empty. However, the intersection of any two is nonempty. What is the smallest possible number of elements contained in their union?

Answer: 14

Solution: Let U be the union of the finite sets. A 12-element set has $2^{12} = 4096$ subsets, therefore, there must be at least 13 elements in U.

Suppose U has 13 elements. We can pair each subset of U with its complement. There are 4096 such pairs. Since there are 5000 sets in the collection, there must be a pair consisting of a set and its complement among the sets in the collection. But these two sets have empty intersection, contradicting the requirement of the problem.

Suppose U has 14 elements. There are

$$\frac{2^{14} - \binom{14}{7}}{2} = 6476$$

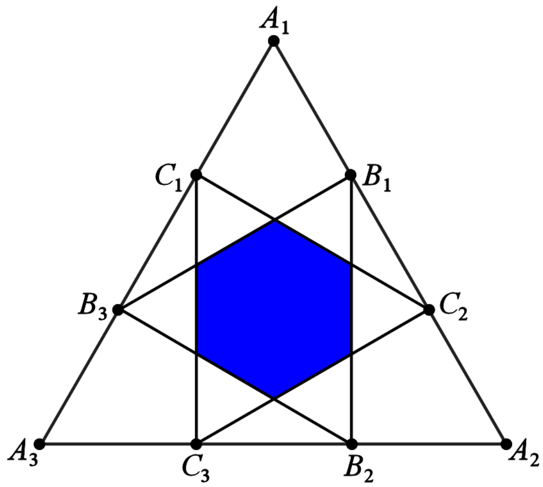
subsets of U which contain 8 or more elements. We create a collection of 5000 subsets of U by taking the 14 subsets that have 13 elements and any $5000 - 14$ other subsets that have at least 8 elements, but not all 14 elements. Then any two sets in this collection will have nonempty intersection because otherwise U would have to have at least 16 elements. Also, the intersection of all 5000 sets is empty because no element is in all 14 of the subsets with 13 elements.

Thus, the answer is $\boxed{14}$.

Problem 4 Let $\triangle A_1A_2A_3$ be an equilateral triangle with unit side length. For $k = 1$, 2, and 3, let B_k be the point on the boundary of $\triangle A_1A_2A_3$ located $1/3$ unit away from A_k in the clockwise direction and let C_k be the point on the boundary of $\triangle A_1A_2A_3$ located $1/3$ unit away from A_k in the counterclockwise direction. What fraction of the area of $\triangle A_1A_2A_3$ is the area of the intersection of $\triangle B_1B_2B_3$ and $\triangle C_1C_2C_3$? Express your answer as a fraction in simplest form.

Answer: $\dfrac{2}{9}$

Solution:



Reflection in any altitude of $\triangle A_1A_2A_3$ swaps $\triangle B_1B_2B_3$ and $\triangle C_1C_2C_3$. Therefore, the desired intersection, indicated in blue is equilateral. Note that $\triangle A_1B_3C_2$ is equilateral since $A_1B_3 = A_1C_2 = 2/3$ and $m\angle B_3A_1C_2 = 60°$. Since B_1 is the midpoint of $\overline{A_1C_2}$, we deduce that $\overline{B_1B_3}$ is perpendicular to $\overline{A_1A_2}$. In a similar way, we see that all the edges of the hexagon are perpendicular to some side of $\triangle A_1A_2A_3$, from which we deduce that the angles of the hexagon all measure 120°. Thus, the intersection is a regular hexagon. The distance between opposites sides of this hexagon is $1/3$. Therefore, the hexagon consists of 6 equilateral triangles each of height $1/6$. The scale factor between any one of these 6 triangles and $\triangle A_1A_2A_3$ is therefore $1/6 : \sqrt{3}/2$, so the ratio of the areas is $1/27$. There are 6 triangles in the hexagon, so the ratio of the area of the intersection to that of $\triangle A_1A_2A_3$ is $6(1/27) = \boxed{2/9}$.

Problem 5 Acute triangle ABC has area 870. The triangle whose vertices are the feet of the altitudes of $\triangle ABC$ has area 48. Determine

$$\sin^2 A + \sin^2 B + \sin^2 C.$$

Express your answer as a fraction in simplest form.

Answer: $\dfrac{298}{145}$

Solution: Let H_A be the foot of the altitude from A, and similarly define H_B and H_C. Note that $\triangle ABC \sim \triangle AH_BH_C$ with scale factor $\cos A$. Similarly, $\triangle ABC \sim \triangle BH_CH_A$ with scale factor $\cos B$ and $\triangle ABC \sim \triangle CH_AH_B$ with scale factor $\cos C$. Therefore, $870 - 48 = 870(\cos^2 A + \cos^2 B + \cos^2 C)$. Thus, $\sin^2 A + \sin^2 B + \sin^2 C = (1 - \cos^2 A) + (1 - \cos^2 B) + (1 - \cos^2 C) = 3 - (1 - 48/870) = \boxed{298/145}$.

Problem 6 Solve for x:

$$\begin{aligned} v - w + x - y + z &= 79 \\ v + w + x + y + z &= -1 \\ v + 2w + 4x + 8y + 16z &= -2 \\ v + 3w + 9x + 27y + 81z &= -1 \\ v + 5w + 25x + 125y + 625z &= 79. \end{aligned}$$

Answer: 24

Solution: Define $p(a) = v + wa + xa^2 + ya^3 + za^4$. The five equations are $p(a) = (a-2)^4 - 2$ for $a = -1$, 1, 2, 3, and 5. Because a 4th degree polynomial is uniquely defined once 5 of its values are specified, it must be that x is the coefficient of a^2 in $p(a) = (a-2)^4 - 2$, which shows that $x = \binom{4}{2}(-2)^2 = \boxed{24}$.

Problem 7 An arithmetic expression is created by inserting either a plus sign or a multiplication sign in each of the 11 spaces between consecutive $\sqrt{3}$'s in a row of twelve $\sqrt{3}$'s. The signs are chosen uniformly and independently at random. What is the probability that the resulting expression evaluates to $12\sqrt{3}$? Express your answer as a fraction in simplest form.

Answer: $\dfrac{15}{512}$

Solution: After the plus signs and multiplication signs are inserted, we obtain a sum of powers of $\sqrt{3}$, which has the form $a + b\sqrt{3}$, where a and b are nonnegative integers. If any power is an even power of $\sqrt{3}$, we would have $a > 0$, which cannot occur. Now observe that $(\sqrt{3})^1 = \sqrt{3}$ and $(\sqrt{3})^3 = 3\sqrt{3}$, but $(\sqrt{3})^{2m+1} > (2m+1)\sqrt{3}$ for $m > 1$ since this inequality is equivalent to $3^m > 2m + 1$ which follows from the binomial theorem applied to $(1 + 2)^m$. Therefore, the only expressions that evaluate to $12\sqrt{3}$ are those which result in a sum of first and third powers of $\sqrt{3}$ only.

In other words, we must count the number of ways to assign the arithmetic symbols so that the multiplication symbols occur only as isolated consecutive pairs.

One approach is to observe that for a fixed number of multiplication symbol pairs, the number of ways to place the pairs can be treated as a "balls in urns" combinatorial problem, where the balls are plus signs and the urns are the spaces before, between, and after pairs of consecutive multiplication symbols.

For 0 pairs, there are 11 balls and 1 urn, yielding 1 expression.

For 1 pair, there are 9 balls and 2 urns, yielding 10 expressions.

For 2 pairs, there are 6 balls (because we must insist that one plus sign goes in between the two pairs of consecutive multiplication signs) and 3 urns, yielding 28 expressions.

For 3 pairs, there are 3 balls and 4 urns, yielding 20 expressions.

For 4 pairs, there are 0 balls and 5 urns, yielding 1 expression.

No further pairs are possible, giving us a grand total of $1+10+28+20+1 = 60$ expressions and a probability of $60/2^{11} = \boxed{15/512}$.

Alternatively, to count the number of sign assignments, note that by appending a plus sign to the end, we are counting the number of ways to domino a 1 by 12 rectangle with 1 by 1 dominoes (which each correspond to a plus sign) and 1 by 3 dominoes (which each correspond to two multiplication signs followed by a plus sign). If c_n is the number of such domino tilings of a 1 by n rectangle, we have $c_{n+3} = c_{n+2} + c_n$ and $c_0 = c_1 = c_2 = 1$. We can then compute the first thirteen terms: 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60.

Problem 8 For a positive integer n, let $p(n)$ denote the number of distinct prime numbers that divide evenly into n. Determine the number of solutions, in positive integers n, to the inequality $\log_4 n \le p(n)$.

Answer: 13

Solution: We split the solution into cases determined by the different possible values of $p(n)$.

Note that there are no solutions for $p(n) = 5$ since the product of the first 5 primes, 2310, exceeds $4^5 = 1024$, and since all primes other than the first two exceed 4, $\log_4 n > p(n)$ for all $p(n) > 5$, as well.

If $p(n) = 0$, then $n = 1$, and 1 is a solution since $\log_4 1 = 0$.

If $p(n) = 1$, our inequality becomes $\log_4 n \le 1$, so $n \le 4$. The numbers less than or equal to 4 that are a power of a prime are: 2, 3, and 4.

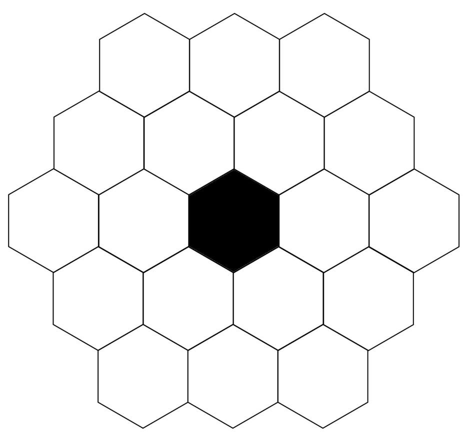
If $p(n) = 2$, our inequality becomes $\log_4 n \le 2$, so $n \le 16$. The numbers less than or equal to 16 that are a product of powers of two distinct prime numbers are: 6, 10, 12, 14, and 15.

If $p(n) = 3$, our inequality becomes $\log_4 n \le 3$, so $n \le 64$. The numbers less than or equal to 64 that are a product of powers of three distinct prime numbers are: 30, 42, and 60.

If $p(n) = 4$, our inequality becomes $\log_4 n \le 4$, so $n \le 256$. The only number less than or equal to 256 that is a product of powers of four distinct prime numbers is: 210.

Thus, there are $\boxed{13}$ solutions.

Problem 9 The ring shown below is made out of 18 congruent regular hexagons. How many ways are there to tile the ring using tiles that consist of two hexagons, each congruent to any one of the 18 in the design, joined edge-to-edge? (The central hexagon, in black, is not to be covered with a tile and the ring cannot be rotated or reflected.)



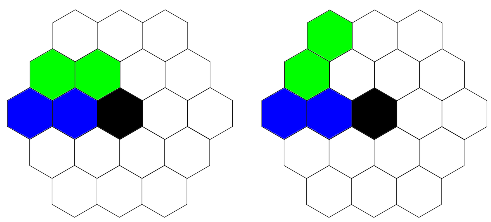
Answer: 104

Solution: Call a tile that covers two hexagons that are split by a radial line that begins at the center of the central, black hexagon, and passes through the midpoint of one of its sides a "spoke." We shall also call the six hexagons that share an edge with the central, black hexagon, the "inner layer" and all 12 remaining non-black hexagons the "outer layer."

We organize our count via the number of spokes.

Notice that no two spokes can share an edge, because that would isolate a single hexagon. Therefore, there can be 0, 1, 2, or 3 spokes, which we handle as four separate cases.

Suppose there are 0 spokes. Then our tiling combines a tiling of the outer layer with a tiling of the inner layer. There are 2 ways to cover each layer with tiles, giving us a total of 4 tilings.



Suppose there is 1 spoke, indicated in blue in the two diagrams above. The hexagon above the blue tile which touches both hexagons of the blue tile can be covered in two ways by a tile, as shown in green in the two figures above. In the left case, there is only 1 way to complete the tiling. (As you complete it, remember that you cannot involve any other spoke!) In the right case, by working out the tree of possibilities, we find that there are 5 ways to complete the tiling. Since there are 6 ways to place the single spoke, this case yields 36 tilings.

Suppose there are 2 spokes. Notice that these spokes cannot lie upon a diameter because diametrically opposed spokes split the remaining part of the ring into two groups each with an odd number of hexagons. Therefore, the 2 spokes must be on radial lines that form a 120° angle, splitting the uncovered parts of the ring into two sections, one with 4 hexagons, the other with 10. Working out the tree of possibilities for each section, we find that the 4-hexagon section can be tiled in 2 ways and the 10-hexagon section can be tiled in 4 ways, giving us a total of 8 tilings for a given placement of the two spokes. Since there are 6 ways to place the two spokes, this case yield 48 tilings.

Finally, suppose there are 3 spokes. These spokes must be evenly spaced, splitting the uncovered hexagons into three sections each consisting of four hexagons, which can each be tiled in 2 ways for a total of 8 tilings for a given arrangement of the three spokes. Since there are 2 ways to place the three spokes, this case gives us 16 tilings.

In total, we have $4 + 36 + 48 + 16 = \boxed{104}$ tilings.

Problem 10 Find all integers x between 0 and the prime number 4099 such that $x^3 - 3$ is divisible by 4099. Write your answer as a list of integers (in any order).

Answer: 520, 3595, 4083

Solution: Note that $4099 - 3 = 4096 = 2^{12}$. So a cube root of -3, modulo 4099, is $2^4 = 16$. If we can find the cube roots of -1, then we can multiply each by 16 to get the cube roots of 3. The cube roots of -1 are the negatives of the cube roots of 1, and the cube roots of 1 are 1 and $(-1 \pm \sqrt{-3})/2$. Now, the square root of -3 is $2^6 = 64$, modulo 4099. Hence, the cube roots of 1 are 1, $63/2$, and $-65/2$. We multiply each by -16 to find that the cube roots of 3 are

$$\boxed{520, 3595, 4083}.$$

Problem 11 A random triangle is produced as follows. A pair of standard dice is rolled independently three times to get three random numbers between 2 and 12, inclusive, by adding the numbers that come up on each pair rolled. Call these three random numbers a, b, and t. The random triangle has two sides of lengths a and b with the angle between them measuring $15(t - 1)$ degrees. What is the probability that the triangle is a right triangle? Express your answer as a fraction in simplest form.

Answer: $\dfrac{113}{648}$

Solution: If $t = 7$, then the triangle is a right triangle. This occurs with probability $1/6$.

If $t > 7$, it is not possible to be a right triangle because a right triangle cannot have an obtuse angle.

If $t < 7$, either the side of length a or the side of length b would have to be the hypotenuse, so that either $a\cos((t-1)15^\circ) = b$ or $b\cos((t-1)15^\circ) = a$, which means that $\cos((t-1)15^\circ)$ must be rational. Of the possible angle measures 15°, 30°, 45°, 60°, and 75°, only $\cos(60^\circ) = 1/2$ is rational, which corresponds to $t = 5$. When $t = 5$, either $a = 2b$ or $b = 2a$. The probability that $t = 5$ is $4/36$. The probability that $a = 2b$ is

$$\frac{3(1) + 5(2) + 5(3) + 3(4) + (1)(5)}{36^2} = \frac{45}{36^2}.$$

This is also the probability that $b = 2a$. Since $a = 2b$ and $b = 2a$ are mutually exclusive, the probability that $t = 5$ and the triangle is right is $2\frac{4}{36}\frac{45}{36^2}$.

We conclude that the desired probability is

$$\frac{1}{6} + 2\frac{4}{36}\frac{45}{36^2} = \boxed{\frac{113}{648}}.$$

Problem 12 Let S be the set of fractions of the form $\frac{\text{lcm}(A,B)}{A+B}$, where A and B are positive integers and $\text{lcm}(A, B)$ is the least common multiple of A and B. What is the smallest number exceeding 3 in S? Express your answer as a fraction in simplest form.

Answer: $\dfrac{52}{17}$

Solution: Let $d = \gcd(A, B)$ and define a and b so that $A = ad$ and $B = bd$. Then $\frac{\text{lcm}(A,B)}{A+B} = \frac{ab}{a+b}$, so without loss of generality, we seek the fraction $\frac{ab}{a+b}$ which is closest to but greater than 3 among relatively prime pairs a and b.

By symmetry, we may also assume $a < b$. (If $a = b$, since a and b are relatively prime, we must have $a = b = 1$, in which case $\frac{ab}{a+b} = 1/2 < 3$.) Note that $\frac{ab}{a+b} = a - \frac{a^2}{a+b}$, which is a strictly increasing function of b when a is held constant. Therefore, for given a, the minimum occurs when $b = a + 1$, and since consecutive integers are relatively prime, $\frac{a(a+1)}{2a+1}$ is in S. Now,

$$\frac{a(a+1)}{2a+1} = \frac{a}{2}\left(1 + \frac{1}{2a+1}\right) < \frac{ab}{a+b} = a - \frac{a^2}{a+b} < a$$

for $b > a + 1$. From this, we see that if $a < 4$, then $ab/(a + b) < 3$ and if $a > 6$, then $ab/(a + b) > 7/2$. If $a = 4$, then the nearest fraction to 3 above 3 occurs when $b = 13$, yielding the fraction $52/17$. If $a = 5$, then the nearest fraction to 3 above 3 occurs when $b = 8$, yielding $40/13$. If $a = 6$, then the nearest fraction to 3 above 3 occurs when $b = 7$, yielding $42/13$. Of these, $52/17$ is closest to 3, and so the answer is $\boxed{52/17}$.

Problem 13 Let
$$f(t) = \frac{(10+9i)t - 10 + 9i}{t+i},$$
where $i = \sqrt{-1}$. Let $P = f(0)$, $Q = f(2023)$, and $R = f(1)$. Determine $\sin^2(m\angle PQR)$. Express your answer as a fraction in simplest form.

Answer: $\frac{1}{2}$

Solution: Note that f is a Möbius transformation, so maps lines to lines or circles in the complex plane. (Alternatively, check that $|f(t) - (\frac{19}{2} + \frac{19}{2}i)|$ is $\frac{\sqrt{2}}{2}$ for t real.) We compute that $P = f(0) = 9 + 10i$ and $R = f(1) = 9 + 9i$. Also, $S \equiv \lim_{t\to\infty} f(t) = 10 + 9i$. So P, R, and S are 3 of the 4 vertices of a unit square, with $\overline{PR}$ being one side of this square, hence the image of $f(t)$ for t real is the circumscribed circle of $\triangle PRS$ (minus S, unless we include the "point at infinity"). By continuity, $Q = f(2023)$ must be a point on this circle on the lower arc whose endpoints are R and S. Thus, the measure of $\angle PQR$ is 45° and $\sin^2(m\angle PQR) = \boxed{1/2}$.

Problem 14 Five points are chosen uniformly and independently at random on the surface of a sphere. Next, 2 of these 5 points are randomly picked, with every pair equally likely. What is the probability that the 2 points are separated by the plane containing the other 3 points? Express your answer as a fraction in simplest form.

Answer: $\frac{2}{5}$

Solution: We claim that the convex hull of the 5 points is a triangular bipyramid. To see this, note that having 4 coplanar points is a 0 probability event, so all the faces of the convex hull are triangles. If E is the number of edges of the convex hull and F is the number of faces, then $E = 3F/2$. Euler's formula tells us that $5 - E + F = 2$, from which we find $F = 6$ and $E = 9$. Thus, the degree of the average vertex is $18/5$, and hence, at least one vertex must have degree greater than 3. Since there are only 5 vertices, there is a vertex with degree 4. Thus, there are 4 triangular faces that share a common vertex A. The 4 vertices other than A form a nonplanar quadrilateral, contributing 2 more faces and 1 more edge, which must connect two opposite vertices of the quadrilateral. The 2 unconnected vertices of the quadrilateral thus have degree 3 and are the apices of triangular pyramids that share the common triangular base formed by the other 3 vertices.

In a convex triangular bipyramid, there are 6 edges emanating from the two degree-3 vertices. If we pick 2 vertices that are endpoints of such an edge, they cannot be split by the plane through the complementary vertices since the complementary vertices are the vertices of a triangular face. However, any 2 vertices of degree 4 must be split be the plane containing the complementary vertices, since if not, then the triangle whose vertices are the complementary vertices would have to be a face, but that's not possible since among those vertices are the two unconnected vertices of degree 3. The only pair of vertices left to consider are the two degree-3 vertices not connected by an edge. These two vertices must be split by the plane containing the complementary vertices since the triangle whose vertices are the complementary vertices is also not a face. Hence, 4 of the 10 pairs of points are split by the plane containing the other points and the answer is $4/10 = \boxed{2/5}$.

Problem 15 A square is divided into four non-overlapping isosceles triangles. Let X be the degree measure of one of the twelve angles of these four triangles. Compute the sum of all possible different values of X. (Consider all possible diagrams.)

Answer: 690

Solution: We say that a shape is *unsplittable* if it cannot be partitioned into the contextually appropriate number of isosceles triangles.

Note that two triangles can be joined along edges to produce: a triangle or a quadrilateral (if joined edge-to-edge), a nonconvex pentagon (if not edge-to-edge, but with a shared vertex), or a nonconvex hexagon (if none of the vertices match up). This implies that a quadrilateral can be partitioned into two triangles only by drawing in one of its diagonals.

Let the square be $ABCD$.

First, suppose that one of the corners of the square is also a corner of an isosceles triangle, which necessarily must be its apex angle. Without loss of generality, we may assume that $\angle BAC$ is the apex angle of an isosceles triangle $\triangle AXY$, with X on $\overline{AB}$ and Y on $\overline{AD}$.

We show that we may assume $X = B$ and $Y = D$ using proof by contradiction. Assume $X \neq B$ and $Y \neq D$. Then $YXBCD$ is a pentagon. Removal of an isosceles corner from any corner of this pentagon results in a convex pentagon or hexagon, with the only exceptions being removal of $\triangle BCD$, $\triangle YXB$, or $\triangle YXD$, with the last two possibilities only possible in the special circumstance where $\angle YXD$ measures $(45/2)°$. Since a convex pentagon or hexagon cannot be built from two triangles, those possibilities are eliminated from consideration. If $\triangle BCD$ is removed, we're left with a trapezoid with 45° and 135° angles, and neither diagonal splits this into two isosceles triangles. If $\triangle YXD$ or $\triangle YXB$ is removed, the remaining quadrilateral is readily seen to be unsplittable. Thus, we may assume without loss of generality that $X = B$ and $Y = D$.

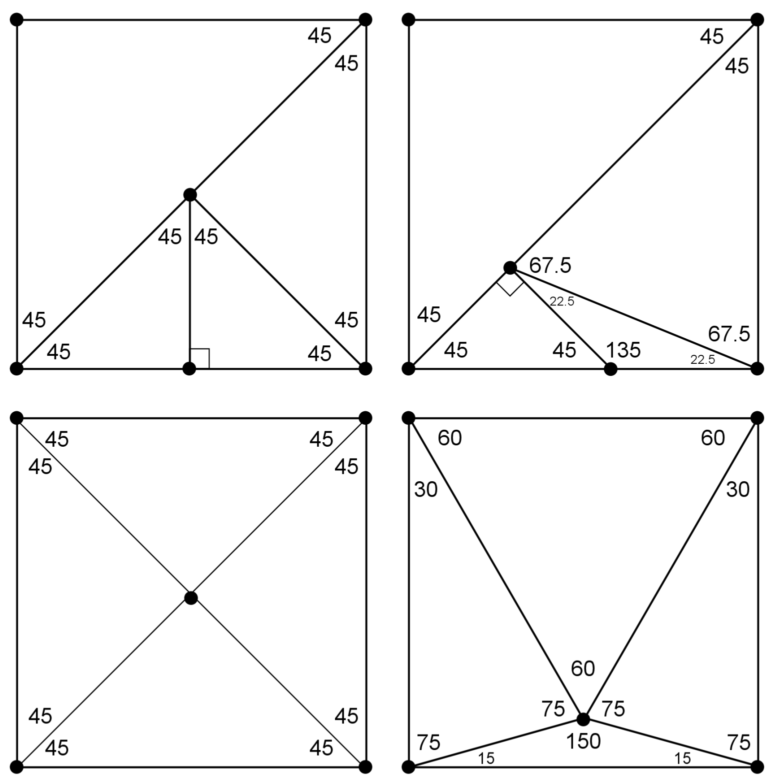
Then, $\triangle BCD$ must be tiled by three isosceles triangles. Now, C cannot be the apex angle of one of the tiles for reasons already seen. If both segments $\overline{BC}$ and $\overline{CD}$ are split by the tiling, that would imply at least 4 tiles contained in $\triangle BCD$, a contradiction. Therefore, we may assume without loss of generality that side $\overline{BC}$ is the side of one of the tiles and label the third vertex of this triangle Z. Note that Z is either on the perpendicular bisector of $\overline{BC}$ or it is on the arc of the circle of radius BC, centered at B. In fact, Z must be on $\overline{BD}$, for otherwise, quadrilateral $BZCD$ would be a nonconvex quadrilateral that must be split into two isosceles triangles by the diagonal $\overline{ZD}$. But if $\triangle BDZ$ is isosceles, then Z must be on diagonal $\overline{AC}$. So, either Z is located at the center of the square, or Z is on $\overline{BD}$ and $BZ = BC$. In the former case, $\triangle CDZ$ is a right isosceles triangle. which can be partitioned into two isosceles triangles in only one way: as the union of two congruent right isosceles triangles. In the latter case, $\triangle CDZ$ has angles that measure 45°, 22.5°, and 112.5°, which can be split into two isosceles triangles in only one way: splitting the angle at Z into a right angle and a 22.5°-angle, forming one isosceles right triangle and one isosceles triangle with an apex angle measuring 135°. See the upper two tilings in the figure.

Next, we assume that the tiling splits all four corner angles of the square. If every corner is split, each side of the square must be part of an isosceles triangle that does not involve any other of the square's sides. Let $\triangle ABX$, $\triangle BCY$, $\triangle CDZ$, and $\triangle DAW$ be these four isosceles triangles, which, therefore, must partition the square. In particular, $\overline{BX}$ and $\overline{BY}$ are on the same line. Without loss of generality, we may assume that X is not farther from B than Y. We claim that $X = Y$.

Suppose to the contrary that X is nearer to B than Y. Now W must be on $\overline{AX}$. If $W \neq X$, then removal of the three triangles $\triangle ABX$, $\triangle BCY$, and $\triangle DAW$ would leave a non-triangle. Therefore $W = X$. Similarly, Y must be on $\overline{DW}$, for otherwise, removal of the three triangles $\triangle ABX$, $\triangle BCY$, and $\triangle DAW$ would leave a quadrilateral. Therefore, both X and Y must be on diagonal $\overline{BD}$, and, consequently, $Y = Z$. But this exhibits right isosceles triangle ABD as a union of two isosceles triangles, which is only possible in one way, namely, with $X = W$ at the midpoint of diagonal $\overline{BD}$. Similarly, $Y = Z$ must be at the midpoint of diagonal $\overline{BD}$. But this means $X = Y$.

By symmetry, we conclude that $X = Y = Z = W$ and the tiling is formed by placing a point interior to the square and drawing line segments from this interior point to each of the corners. We have already seen that it is possible to place X in the center of the square (see the lower left tiling in the figure), so suppose that X is not in the center. Without loss of generality, suppose X is in the quadrant of the square that contains D. Then $DX < DA = DC$. We cannot have both $AX = AD$ and $CX = CD$ since X is interior to the square. Without loss of generality, assume $CX \neq CD$. Then $DX = CX$, which in turn means $AX = BX$. Since X is not central, we must also have $AX = AD$ and $BX = BC$, which implies that $\triangle ABX$ is equilateral (see the lower right tiling in the figure).

Reviewing the four tilings, the angles involved are: 15°, 22.5°, 30°, 45°, 60°, 67.5°, 75°, 90°, 135°, and 150°. The sum of these angles is $\boxed{690}$.



Problem 16 Let $f(x) = x^2 - 3/4$. Let $f^{(n)}(x)$ denote the composition of f with itself n times. For example, $f^{(3)}(x) = f(f(f(x)))$. Let R be the set of complex numbers that is the union of the roots of the polynomials $f^{(n)}(x^2 + 3/4)$ over positive integers n. Let B be the smallest rectangle in the complex plane with sides parallel to the real and imaginary axes that contains R. What is the square of the area of B?

Answer: 27

Solution: For positive integers n, note that

$$f^{(n)}(x^2 + 3/4) = f^{(n)}(-((ix)^2 - 3/4)) = f^{(n)}(-f(ix)) = f^{(n)}(f(ix)).$$

Hence, R is a $90°$ rotation of S, which we define to be the union of the roots of $f^{(n)}(x)$ for $n > 1$. Because $f(x) = f(-x)$ and $f(x)$ has real coefficients, both R and S have mirror symmetry about the real and the imaginary axes.

Because $f(se^{ia}) = s^2e^{2ia} - 3/4$, the circle of radius s centered at the origin in the complex plane is mapped by f to the circle of radius s^2 centered at $-3/4$. By the triangle inequality, $s^2 = |f(se^{ia}) - (-3/4)| \leq |f(se^{ia})| + 3/4$, hence $|f(se^{ia})| \geq s^2 - 3/4$. If $s \geq 3/2$, then $s^2 - 3/4 \geq 3/2$. Therefore R is contained within the circle of radius $3/2$ centered at the origin.

Now observe that $r_n \equiv \sqrt{3/4 + \sqrt{3/4 + \sqrt{3/4 + \cdots + \sqrt{3/4}}}}$ (n occurrences of $3/4$) is a root of $f^{(n)}(x)$. Since the limit of r_n as n tends to infinity is $3/2$, we conclude that the vertical line of complex numbers with real part $3/2$ bounds S on its right. By symmetry, S is bounded on its left by the vertical line of complex numbers with real part $-3/2$.

In a similar manner, by looking at the limit of the roots $s_n \equiv \sqrt{3/4 - r_n}$, we deduce that the horizontal bounding lines of S correspond to complex numbers with imaginary part $\pm c$ for some $c \geq \sqrt{3}/2$. We claim that $c = \sqrt{3}/2$. To see this, note that $f(t + \sqrt{3}i/2) = (t^2 - 3/2) + \sqrt{3}ti$. For real t, this parametrizes a parabola with apex at $-3/2$ and opening to the right. Points on this parabola have distance $\sqrt{(t^2 - 3/2)^2 + (\sqrt{3}t)^2} = \sqrt{t^4 + 9/4}$ from the origin, and, hence, are not contained inside the circle of radius $3/2$ centered at the origin. Complex numbers with imaginary part greater than $\sqrt{3}/2$ are mapped to the *left* of this parabola. Therefore, no complex number in S is above the line of complex numbers with imaginary part $\sqrt{3}/2$ or below the line of complex numbers with imaginary part $-\sqrt{3}/2$.

We conclude that the answer is $(3\sqrt{3})^2 = \boxed{27}$.

Problem 17 Let C be a unit cube. Let D be a translate of C such that one corner of D is located at the center of C and one corner of C is located at the center of D. Let D' be the image of D under a 60° clockwise rotation about the line that passes through both cube centers when looking from the center of D to the center of C. What is the volume of the intersection of C with D'? Express your answer as a fraction in simplest form.

Answer: $\dfrac{9}{64}$

Solution: Notice that the line through the cube centers passes through a major diagonal of both cubes. If we project the edges of C that emanate from the center of D and the edges of D that emanate from the center of C onto the plane perpendicular to this line, we get 6 spokes, which, by symmetry, are evenly spaced. A 60° clockwise rotation brings the projection of these edges of D into alignment with the projection of these edges of C, and so the lines containing these edges must intersect (in space). By symmetry, they must intersect in the plane that is the perpendicular bisectors of the centers of the two cubes. Since $1/4 < 1/3$, this plane in fact intersects the relevant edges in their interiors, $\frac{1/4}{1/3} = 3/4$ of the way down each edge from their common point of intersection. Therefore, the intersection of C and D' is the union of two equilateral triangular pyramids joined along their base. The height of each pyramid is half the distance between the cube centers, which is $\sqrt{3}/4$ and the base is an equilateral triangles with side length $3\sqrt{2}/4$. Such a pyramid has volume $\frac{1}{3}(\frac{\sqrt{3}}{4}(3\sqrt{2}/4)^2)\frac{\sqrt{3}}{4} = 9/128$. The volume of the intersection is twice this or $\boxed{9/64}$.

Problem 18 A unit square is decorated with snippets of the graph of $y = x^2$ as follows: We consider the graph of $y = x^2$ restricted to the domain $0 \le x \le 6$. We cut up the first quadrant (the positive quadrant) into unit squares with lattice vertices. We translate each square so that they are stacked, one on top of the other. We merge all of these squares. How many regions is the unit square divided into by all the overlaid snippets of the graph of the parabola?

Answer: 78

Solution: Let $N(n)$ be the number of regions created in the manner described in the problem statement for x confined to the domain $[0, n]$. Let k be a nonnegative integer. The values of x in $[\sqrt{k}, \sqrt{k+1}]$ correspond to moving from the bottom to the top of the unit square exactly once, and these strands, for $k = n^2, \ldots, (n+1)^2 - 1$, do not intersect in their interiors. We can determine $N(n+1)$ by adding to $N(n)$ the number of new regions formed by these strands. The number of new regions formed by each strand is 1 plus the number of times the strand intersects with the *interiors* of strands corresponding to x in $(\sqrt{l}, \sqrt{l+1})$ for integers $l < n^2$.

Before proceeding with the inductive computation of $N(6)$, let us determine all the possible intersections between strand interiors. In other words, for $0 < x < 1$, when are there integers m and n such that $(x+m)^2 - (x+n)^2$ is an integer? This occurs when $2x(m-n)$ is an integer. Let $d = m - n$.

If $d = 1$, then $x = 1/2$. (Thus, all numbers whose fractional part is $1/2$ have a square with fractional part $1/4$.)

When $d = 2$, then x is a multiple of $1/4$. However, we already understand what happens when x is a multiple of $1/2$, so the only new points of intersection occur over $x = 1/4$ and $x = 3/4$.

We construct a table row by row where the first column is d and the second column gives values of x where strand interiors may intersect for such values of d, but not smaller:

d	x
1	$1/2$
2	$1/4, 3/4$
3	$1/6, 2/6, 4/6, 5/6$
4	$1/8, 3/8, 5/8, 7/8$
5	$1/10, 2/10, 3/10, 4/10, 6/10, 7/10, 8/10, 9/10$

Let $L(d)$ be the number of entries in the second column of row d in the table above. (Note that for $d > 1$, we have $L(d) = 2\phi(d)$, where ϕ is the Euler ϕ-function.) Thus, $N(n+1) - N(n) = (n+1)^2 - n^2 + \sum_{k=1}^{n} L(k)$.

We now inductively compute $N(n)$, beginning with $N(1) = 2$:

n	$N(n) - N(n-1)$	$N(n)$
1		2
2	$(2^2 - 1^2) + 1$	6
3	$(3^2 - 2^2) + 3$	14
4	$(4^2 - 3^2) + 7$	28
5	$(5^2 - 4^2) + 11$	48
6	$(6^2 - 5^2) + 19$	78

Thus, the answer is $\boxed{78}$.

Problem 19 Let $N = \prod_{k=1}^{1000}(4^k - 1)$. Determine the largest positive integer n such that 5^n divides evenly into N.

Answer: 624

Solution: Let $v_5(n)$ denote the largest k for which 5^k divides evenly into n.

Thinking of 4 as $5 - 1$, we have

$$4^n - 1 = 5^n - \binom{n}{1}5^{n-1} + \binom{n}{2}5^{n-2} - \cdots + (-1)^{n-1}\binom{n}{n-1}5 + (-1)^n - 1.$$

Since $(-1)^n - 1$ equals 0 or -2 depending on whether n is even or odd, respectively, we see that $4^n - 1$ is divisible by 5 if and only if n is even.

Suppose n is even. We claim $v_5(\binom{n}{k}5^{n-k}) > v_5(5n)$, for $0 \le k < n-1$. This is equivalent to showing that $v_5(\binom{n}{k}5^k) > v_5(5n)$, for $1 < k \le n$. Let $m = v_5(n)$. (Note that $v_5(5n) = 1 + m$.) Since $n \le 1000$, we know that $m < 5$. If $k > 1 + m$, there is nothing to prove, so assume $1 < k \le 1 + m$.

Using Kummer's theorem, we deduce the following facts: If $1 < k < 5$, then $v_5(\binom{n}{k}) \ge m$, hence $v_5(\binom{n}{k}5^k) \ge m + k > m + 1$. If $k = 5$ and $m > 0$, then $v_5(\binom{n}{5}) \ge m - 1$ and so $v_5(\binom{n}{5}5^5) \ge m - 1 + 5 = m + 4 > m + 1$. Finally, if $k = 5$ and $m = 0$, then $v_5(\binom{n}{5}5^5) \ge 5 > 1$.

Therefore, for n even, we have

$$4^n - 1 = -5n + (\text{a multiple of } 5^{2+m}),$$

and hence, $v_5(4^n - 1) = 1 + m$.

Hence, $v_5(N) = 1000/2 + 1000/10 + 1000/50 + 1000/250 = \boxed{624}$.

Problem 20 Let $f_1(x) = 2\pi \sin(x)$. For $n > 1$, define $f_n(x)$ recursively by

$$f_n(x) = 2\pi \sin(f_{n-1}(x)).$$

How many intervals $[a, b]$ are there such that

- $0 \le a < b \le 2\pi$,
- $f_6(a) = -2\pi$,
- $f_6(b) = 2\pi$,
- and f_6 is increasing on $[a, b]$?

Answer: 682

Solution: We shall call a segment of a graph that strictly increases from 0 to 2π, then strictly decreases from 2π back to 0, a "hump," and a segment of a graph that strictly decreases from 0 to -2π, then strictly increases from -2π back to 0, a "trough."

We claim that between consecutive 0's, the function f_n is either a hump or a trough. We shall prove this by induction on n, where the base case $n = 1$ is clear. Suppose our claim is true for $n = N$. Let $z_1 < z_2 < z_3 < \cdots < z_m$ be the zeroes of f_N in order. (Note that $z_1 = 0$ and $z_m = 2\pi$.) On the interval $[z_k, z_{k+1}]$, the function f_N, by induction, is a hump or a trough. Therefore on $[z_k, z_{k+1}]$, f_{N+1} either goes (always strictly monotonically) from 0 to 2π back to 0, then to -2π, then back to 0, then back to -2π, then to 0, then to 2π, then back to 0, or does the negative of this, depending on whether f_N is a hump or a trough, respectively. Either way, the property we claim is true. Specifically, on $[z_k, z_{k+1}]$, if f_N is a hump, then f_{N+1} consists of a hump, followed by two troughs, followed by a hump, and if f_N is a trough, then f_{N+1} consists of a trough, followed by two humps, followed by a trough.

An interval of the desired type occurs precisely when a trough is followed directly by a hump. Let B_n be the number of humps or troughs of f_n and let I_n be the number of troughs that are followed directly by a hump for f_n. The inductive analysis shows that $B_{n+1} = 4B_n$. Also, a trough followed directly by a hump in f_n corresponds to a trough followed by a hump in f_{n+1} with the same zero. In addition, each hump or trough of f_n yields one new trough that is directly followed by a hump in f_{n+1} that introduces a new zero. That is, $I_{n+1} = I_n + B_n$. From $B_{n+1} = 4B_n$ and $B_1 = 2$, we see that $B_n = 2(4^{n-1})$. Thus, $I_{n+1} - I_n = 2(4^{n-1})$. Summing these equations for $n = 1, \ldots, N-1$, we have $I_N - I_1 = 2(1 + 4 + 4^2 + \cdots + 4^{N-2})$. Since $I_1 = 0$, we find $I_N = (4^N - 4)/6$. Therefore, the answer is $I_6 = \boxed{682}$.