Math Prize for Girls Olympiad

Advantage Testing Foundation/Jane Street

Time Limit: 4 hours

Thursday, December 5, 2024

1. Let ABC be an acute scalene triangle and let D be a point in its interior such that $\angle DBA = \angle DCA$. Points E and F lie inside segments AC and AB such that $DE = DC$ and $DB = DF$. Is it possible that lines AD, BE, CF are concurrent?

   (A triangle is *scalene* if none of its three side lengths are equal.)

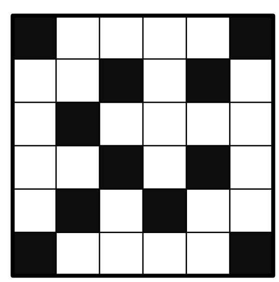
2. Let n and k be positive integers with $10^{k-1} \le n < 10^k$, and let $N = 10^k \cdot (n+1) + n$. (For example, if $n = 2024$, then $k = 4$ and $N = 20252024$.)
   (a) Show that if k is odd, then N is not a perfect square.
   (b) Is it possible that N is a perfect square greater than 10^{2024}?

3. Let ABC be a lattice triangle. Prove that there exist integers m and n such that

$$m^2 + n^2 = (AB \cdot AC)^2 + (BC \cdot BA)^2 + (CA \cdot CB)^2.$$

   (A *lattice point* is a point whose x-coordinate and y-coordinate are both integers. A *lattice triangle* is a triangle whose vertices are lattice points.)

4. Let $n \ge 3$ be an integer and consider an $n \times n$ grid of unit squares. Elaine wishes to paint each of the n^2 squares of the grid either white or black such that the following three conditions all hold:
   (i) Of the $4n-4$ squares touching the edge of the grid, the four corners are black and the other $4n-8$ squares are white.
   (ii) No two black squares share an edge.
   (iii) For every two distinct white squares w and w', there is exactly one white path starting with w and ending with w'.



   Here, a *white path* refers to a finite sequence $(w_0, w_1, \ldots, w_m)$ of distinct white cells such that w_i and w_{i+1} share an edge for all $0 \le i \le m-1$. One example of such a coloring when $n = 6$ is shown above.

   Prove that there are infinitely many integers n for which Elaine can accomplish the task, and infinitely many for which she cannot.