Solutions to Math Prize Olympiad 2024

Advantage Testing Foundation / Jane Street

Fall/Winter 2024

§1 Problem 1, proposed by Anant Mudgal

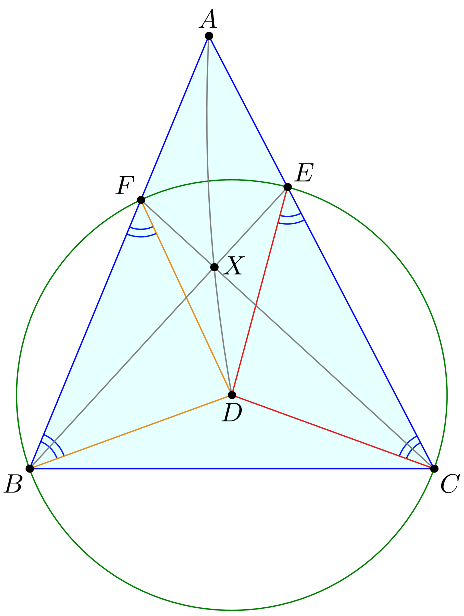
The answer is no, the lines are never concurrent. We proceed by contradiction — showing that the concurrence would imply that $AB = AC$. Refer to the figure below (which is drawn slightly not to scale).

The conditions in the problem imply that we have four equal angles

$$\measuredangle BFD = \measuredangle DBF = \measuredangle DEC = \measuredangle ECD$$

owing to the two (similar) isosceles triangles $\triangle BDF$ and $\triangle DEC$. In particular,

$$\measuredangle BFD = \measuredangle ACD \implies AFDC \text{ is cyclic}$$
$$\measuredangle DEC = \measuredangle DBA \implies AEDB \text{ is cyclic.}$$



Now assume the three lines AD, BE, CF were concurrent at a point X. Then by considering power of a point as

$$BX \cdot XE = AX \cdot XD = CX \cdot XF$$

we obtain that quadrilateral $BFEC$ is cyclic too. Since $DB = DF$ and $DE = DC$ and $BF \not\parallel EC$, it follows that D is the center of that circle. In other words

$$DB = DF = DE = DC.$$

Hence, the two similar isosceles triangles from earlier, $\triangle BDF$ and $\triangle CDE$, are actually congruent. In particular, $BFEC$ would be an isosceles trapezoid, implying $AB = AC$.

§2 Problem 2, proposed by Max Lu and Max Xu

For both parts, it will be easier to work with $N = (10^k + 1) \cdot (n+1) - 1$.

¶ Solution to (a). The idea is to take modulo 11. Note that if k is odd, then

$$N = \underbrace{(10^k + 1)}_{\equiv 0 \bmod 11} \cdot (n+1) - 1 \equiv -1 \pmod{11}.$$

Since no perfect square is $-1 \pmod{11}$, the conclusion follows.

¶ Solution to (b). We prove in fact N can be a perfect square for any even $k \geq 2$. Choose the k-digit number

$$n = (10^{k/2} - 1)^2 + 1.$$

Then one can verify the identity

$$N = \left(10^k + 1\right)\left((10^{k/2} - 1)^2 + 2\right) - 1 = \left(10^k - 10^{k/2} + 1\right)^2.$$

In particular, taking $k = 1014$ finishes.

Remark. At first, the solution to (b) may appear to be a "magical identity". However, it turns out that it actually can be discovered naturally from the solution to (a) using only number theory, without any algebra.

For concreteness, let's show how one would find n when $k = 4$. Then we are trying to find integer solutions to

$$10001(n+1) - 1 = c^2.$$

The idea is that n is largely irrelevant and this just asks

$$c^2 \equiv -1 \pmod{10001}.$$

In part (a), we saw that the issue with odd k was that -1 was not a quadratic residue modulo $10^k + 1$, because it is divisible by 11. So is -1 a quadratic residue modulo 10001? The answer is "obviously yes", because one can take $c \equiv 100 \pmod{10001}$.

This does not work right away since it gives $n = 0$; that is, c is too small. But this can be mitigated by instead choosing the *other* square root modulo 1001:

$$c \equiv -100 \pmod{10001} \iff c \equiv 9901 \pmod{10001}.$$

In other words, $9901^2 + 1$ should be a suitably large multiple of 10001. And indeed, $9901^2 = 98029801$ as we need.

The general situation is the same except one works modulo $10^k + 1$. We know -1 is a quadratic residue because $(10^{k/2})^2 \equiv -1 \pmod{10^k + 1}$ and so one can just take the other square root $(10^k + 1) - 10^{k/2}$ as before. In general, every value of c such that $c^2 \equiv -1 \pmod{10^k + 1}$ and $c > \sqrt{10} \cdot 10^{k-1}$ will give a valid square. For example, in the case $k = 4$, the factorization $10^4 + 1 = 73 \cdot 137$ together with the Chinese remainder theorem can be used to find $82428241 = 9079^2$.

Remark. Amusingly, the submission email for this problem mentioned that it was inspired by "Max's favorite number 8281", but does not specify which Max.

> **Remark.** Although it was proposed by the authors independently, it turned out that this problem has also been previously discussed by Florian Luca and Pantelimon Stănică in https://doi.org/10.1080/00029890.2019.1632628.

§3 Problem 3, proposed by Karthik Vedula

Let $a = BC$, $b = CA$, $c = AB$, for brevity. Note that a^2, b^2, c^2 are integers by the Pythagorean theorem.

By Pick's theorem, it follows that ABC has area half an integer, say $k/2$. However, Heron's formula on the triangle ABC gives an explicit formula for this area:

$$\frac{k}{2} = \frac{1}{4}\sqrt{2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)}$$

This implies that

$$(2k)^2 + (a^4 + b^4 + c^4) = 2(a^2b^2 + b^2c^2 + c^2a^2)$$

or

$$(2k)^2 + (a^2 + b^2 + c^2)^2 = 4(a^2b^2 + b^2c^2 + c^2a^2).$$

In particular, $a^2 + b^2 + c^2$ is an even integer. Dividing by four gives

$$k^2 + \left(\frac{a^2 + b^2 + c^2}{2}\right)^2 = a^2b^2 + b^2c^2 + c^2a^2$$

and this solves the problem.

> **Remark.** This geometric insight shows that in principle, one can find a pure-coordinate solution to the problem using a suitable identity. Specifically, suppose we impose coordinates $A = (0, 0)$, $B = (u, v)$, and $C = (s, t)$. Then $k = |ut - sv|$ and
>
> $$\begin{aligned} a^2 &= (u - s)^2 + (v - t)^2 \\ b^2 &= s^2 + t^2 \\ c^2 &= u^2 + v^2. \end{aligned}$$
>
> Then, if one is willing to check identities by manual expansion, this gives a "one-line proof"
>
> $$a^2b^2 + b^2c^2 + c^2a^2 = (ut - vs)^2 + \left(\frac{(u - s)^2 + (v - t)^2 + u^2 + v^2 + s^2 + t^2}{2}\right)^2.$$
>
> However, it does not seem there is any obvious way to discover this identity without the underlying geometric context.

§4 Problem 4, proposed by Bobby Shen

We show that the task is always impossible if $n \equiv 1 \pmod 3$, but give a dyadic construction for several n.

In both parts, it will be useful to discuss the set of white cells with graph-theoretic language — we consider white cells to be vertices where two cells sharing a side are adjacent. Then condition (iii) is stating that the graph on white cells has a unique path between any two vertices — i.e, that it is a *tree* (equivalently, it is *connected* and *acyclic*). We will use such terminology freely without referring to the underlying graph explicitly.

¶ **Proof of impossibility.** Suppose there are a white squares and b black squares, so $a+b=n^2$. The idea is to double-count unordered pairs of adjacent squares in two ways.

On the one hand, the total number of pairs is obviously $2n(n-1)$.

On the other hand, we can decompose the count by colors:

- There are $\boxed{0}$ black-black pairs of adjacent squares.
- The set of white cells forms a *tree*, so the number of white-white pairs is exactly $\boxed{a-1}$.
- The number of white-black pairs can be counted by noting that every black square is adjacent to exactly 4 white squares, *except* for the four corner black squares, which are adjacent to exactly 2. This gives a total count of $2 \cdot 4 + 4 \cdot (b-4) = \boxed{4b-8}$.

These three numbers need to sum to $2n(n-1)$.

So, in order for the task to be possible with a white and b black squares, it would be necessary (although not obviously sufficient) that

$$a + b = n^2$$
$$(a-1) + (4b-8) = 2n(n-1).$$

At this point, one should suspect that for many values of n this equation has no integer solutions (a, b) at all. And indeed, taking the equations modulo 3 gives

$$n^2 = a + b \equiv a + 4b \equiv 2n^2 + 2n + 8 \pmod 3$$

which is false if $n \equiv 1 \pmod 3$. Hence, Elaine's task is not possible at all if $n \equiv 1 \pmod 3$.

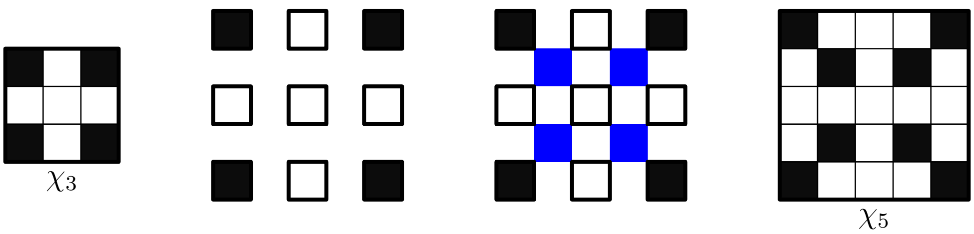
¶ **Recipe for enlarging a satisfying coloring.** In general, a satisfying coloring can be seen as a function $\chi_n \colon \{0, 1, \dots, n-1\}^2 \to \{\mathsf{black}, \mathsf{white}\}$. We show how to take a satisfying coloring χ_n and extend it to a satisfying coloring on χ_{2n-1}. This will let us provide a construction for infinitely many n by induction, starting from any base case.

Claim — If χ_n is any satisfying coloring of an $n \times n$ board for $n \geq 3$, then

$$\chi_{2n-1}\left((x,y)\right) := \begin{cases} \mathsf{black} & \text{if } x \text{ and } y \text{ are both odd} \\ \mathsf{white} & \text{if } x+y \text{ is odd} \\ \chi_n\left((x/2, y/2)\right) & \text{if } x \text{ and } y \text{ are both even.} \end{cases}$$

is a satisfying coloring of a $(2n-1) \times (2n-1)$ board.

We show an example of this procedure below, transforming an $n = 3$ construction into one for $n = 5$. A natural-language description is that χ_{2n-1} consists of a "doubled" copy of χ_n, together with "new" black squares (colored blue below) on the intersection of "gapped" rows and columns.



Proof of claim. Condition (i) holds for χ_{2n-1} since it holds for χ_n. Condition (ii) certainly holds because all black cells have even coordinate sum, so no two can be adjacent. As for condition (iii), the set of white cells is still connected because "old" white cells that were adjacent in χ_n are now connected by an intermediate "new" white cell. And it is still acyclic — if there was a cycle of white cells in χ_n, it must alternate between old and new white ones, in which case taking the old white cells would give a cycle in χ_n, but χ_n was assumed to be satisfying so this can never occur. □

To complete the problem, it remains to take a single valid example (such as the 6×6 coloring in the problem statement) and repeatedly apply this procedure to generate infinitely many n for which the task can be completed.

> **Remark.** The 6×6 example from the statement is obviously not the most natural base case; the simplest construction is a 3×3 square with only four black corners. If one applies the process above, one gets the following patterns shown below for $n = 3$, $n = 5$, $n = 9$, respectively.
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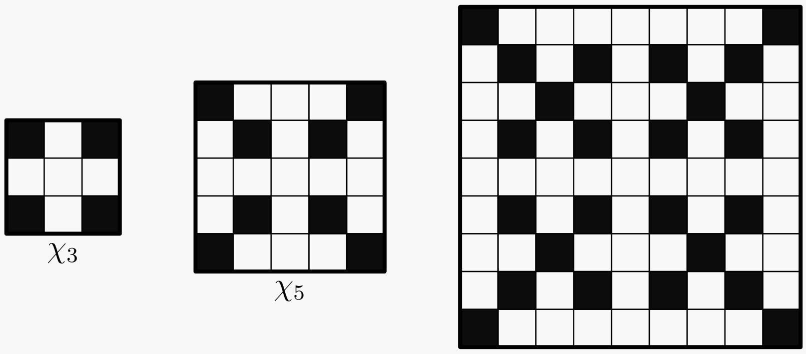
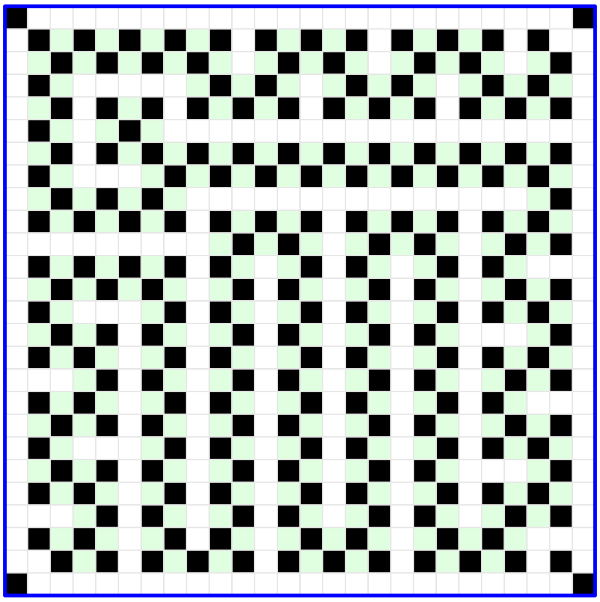
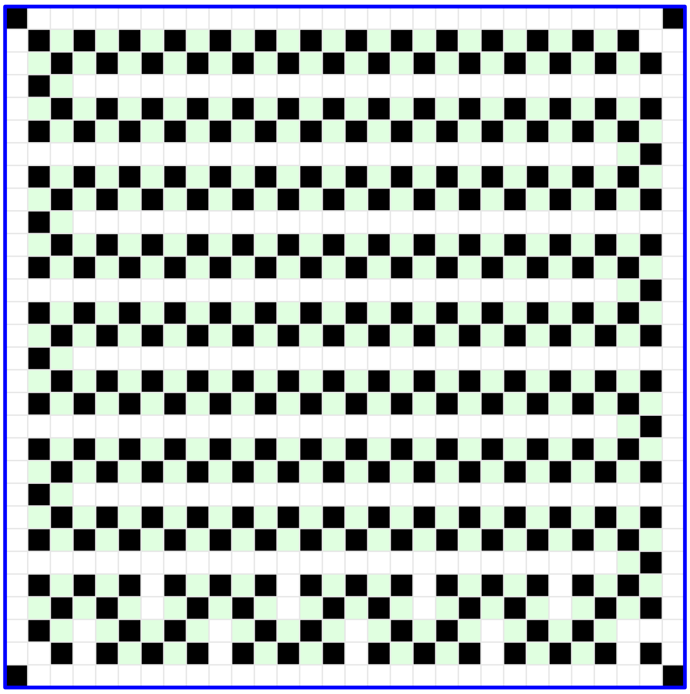
> 
>
> χ_3 χ_5 χ_9
>
> This family of resulting colorings for $n = 2^k + 1$ can actually be described concretely without induction. If we instead the cells by ordered pairs $(x, y) \in \{-2^k, -2^k + 1, \dots, 2^k\}^2$ in the obvious way, then one can encode it as:
>
> $$(x,y) \mapsto \begin{cases} \textsf{white} & \text{if } x = 0 \text{ or } y = 0 \\ \textsf{black} & \text{if } xy \neq 0 \text{ and } \nu_2(x) = \nu_2(y) \\ \textsf{white} & \text{if } xy \neq 0 \text{ and } \nu_2(x) \neq \nu_2(y). \end{cases}$$
>
> While this is simpler to state, it seems harder to verify the condition (iii) when written this way, since one still needs to use some form of induction. Moreover, the general recipe can use any "initial" base case as the coloring, so we chose to present that one first.

¶ Extension: Sketch of all construction for all $n \not\equiv 1 \pmod 3$. The author mentions that Elaine's task is in fact possible for all $n \geq 3$ satisfying $n \not\equiv 1 \pmod 3$, although the construction is not easy to describe in words, and much harder to find than the doubling recipe just mentioned before.

Illustrated below are constructions for $n = 26$ and $n = 30$. These generalize to families of constructions valid for all even $n \geq 12$ with either $n \equiv 2 \pmod 6$ and $n \equiv 0 \pmod 6$, respectively. To improve readability, in the white cells, a light green tint has been added to the leaves of the white tree (i.e. to white cells which are adjacent to only one other white cell).



$n = 26$, extends to all $n \equiv 2 \pmod 6$ for $n \geq 14$



$n = 30$, extends to all $n \equiv 0 \pmod 6$ for $n \geq 12$

This shows that all even $n \geq 12$ with $n \not\equiv 1 \pmod 3$ are possible. The example for $n = 6$ was given already, and one can manually find an example for $n = 8$ as well.

Thus all even $n \geq 6$ are handled. Finally, the previous doubling recipe (going from n to $2n - 1$) inductively produces constructions for all valid odd $n \geq 5$ (besides the case $n = 2^k + 1$ which we already did).