The Sixteenth Annual Math Prize for Girls

Sunday, October 6, 2024

Test Version A

DIRECTIONS

1. Do not open this test until your proctor instructs you to.
2. Fill out the top of your answer sheet. Your student number is on your name tag. Your test version (A, B, C, or D) is above on this page.
3. You may use pencils, pens, and erasers. You may also use the scratch paper that we provide. You may NOT use calculators, rulers, protractors, compasses, graph paper, or books.
4. Write your final answers on the answer sheet. When a problem specifies a particular form for the answer, write your answer in that form. Write your answers clearly. We will collect only the answer sheet from you.

NOTES

1. This test contains 20 problems. You will have 150 minutes (2.5 hours) to take the test. Your score will be the number of correct answers.
2. Figures are not necessarily drawn to scale.
3. Good luck!

1. The lengths of the sides of a (nondegenerate) triangle are consecutive even perfect squares. What is the smallest possible perimeter of the triangle?

2. In square $ABCD$, an ant travels from A to the opposite vertex C along a zigzag path that starts at A, goes straight to P, then straight to Q, then straight to C, where $AP = PQ = QC = CD$ and $m\angle APQ = m\angle PQC$. What is $\cos(m\angle APQ)$? Express your answer as a fraction in simplest form.

3. How many positive integers divide evenly into $19^8 - 2^{12}$?

4. You have a rectangular grid of squares, 3 columns wide and 6 rows high. How many ways are there to color half of its 18 squares black and the other half white so that no two rows have the exact same pattern of black and white squares?

5. The graph of a cubic polynomial has a local maximum at $(-10, 10)$ and a local minimum at $(10, -10)$. What is its leading coefficient? Express your answer as a fraction in simplest form.

6. Let C be a circle with radius 1 and center O. Let A and B be the endpoints of a 90° arc on the circumference of C. The circumference of a circle D, with center Q, intersects the circumference of C at points A and B in right angles. Let P be one of the points on the circumference of D such that $\overline{OP}$ forms a 60° angle with the tangent line to D at P. Determine $\cot^2(m\angle POQ)$.

7. In how many ways can an isosceles right triangle with legs of length 5 be tiled by isosceles right triangles with legs of length 1?

8. Let C be a circle. Let S be the set of points in C that are the centroids of obtuse triangles inscribed in C. What fraction of the area of C is occupied by S? Express your answer as a fraction in simplest form. (The centroid of a triangle, also known as the center of mass, is the point where the three medians intersect.)

9. Six points are chosen uniformly and independently at random on the boundary of an equilateral triangle with perimeter P. ("Uniformly" means that the probability that a point lies on a particular segment of the boundary of length L is equal to L/P.) What is the probability that the convex hull of the 6 points is a quadrilateral? Express your answer as a fraction in simplest form.

10. Let F_n be the Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for integer $n > 1$. How many integers have the form $F_a - F_b$ where $1 \le b < a \le 30$?

11. The quartic polynomial p has integer coefficients and 4 distinct positive integer roots. If $p(4) = -256$ and $p(5) = -135$, what are its roots? Write your answer as a list of numbers in increasing order separated by commas.

12. A circle of radius r is drawn in the xy-coordinate plane centered at the point $(r, 0)$. An ant walks a zigzag path that begins at the origin with slope m. Each time the ant meets either the circumference of the circle or the x-axis, the ant changes direction by negating the slope of the line it is on. The ant always drifts to the right. The ant reaches $(2r, 0)$ after creating a zigzag path with 10 segments. If the first segment of the ant's journey is 1 unit long, what is the total distance that the ant traveled? Write your answer in the form $a + b\sqrt{c}$, where a, b, and c are positive integers and c is square-free.

13. Let p be the unique polynomial of degree 6 such that

$$p(n) = (-1)^n \binom{6}{n}$$

for $n = 0, 1, 2, 3, \ldots, 6$. What is $p(7)$?

14. The set S contains 2024 elements. What is the size of a largest collection C of subsets of S with the property that the intersection of every three subsets in C is nonempty, but the intersection of every four is empty?

15. Let $\triangle ABC$ be equilateral with side length 999. Point P is in the interior of $\overline{BC}$. The distances from P to A, B, and C are all integers. What is AP?

16. For every positive integer n, let d_n be the greatest common divisor of $n^2 + 1$ and $n^2 + n + 10$. Let m be the maximum value attained by the sequence d_n. Let k be the smallest positive integer such that $d_k = m$. What is k?

17. Define $f_n(x)$ recursively by setting $f_0(x) = |x|$ and letting

$$f_n(x) = |n - f_{n-1}(x)|$$

for every integer $n > 0$. Let c be the unique positive integer such that $f_{100}(c) = 0$. What is the area sandwiched between the x-axis and the graph of $y = f_{100}(x)$ for $-c < x < c$?

18. Consider the 24 points in four-dimensional space whose coordinates are the various permutations of 1, 2, 3, and 4. Among these 24 points, how many subsets of 4 form the vertices of a square?

19. The sequence a_n is defined recursively as follows: $a_1 = \tan(15^\circ/2)$ and

$$a_{n+1} = \frac{1 - 2a_n - a_n^2}{1 + 2a_n - a_n^2},$$

for every integer $n \geq 1$. Determine the number A such that $-90 < A < 90$ and $a_{100} = \tan A^\circ$.

20. Let ABC be a triangle with $\angle ACB$ right. Let M be the midpoint of the hypotenuse. Let O be the foot of the altitude from C. Let N be the intersection of the angle bisector at C and the hypotenuse. Given that $MN = 53$ and $NO = 45$, determine the length of the hypotenuse.