Math Prize for Girls

2024 Solutions

Problem 1 The lengths of the sides of a (nondegenerate) triangle are consecutive even perfect squares. What is the smallest possible perimeter of the triangle?

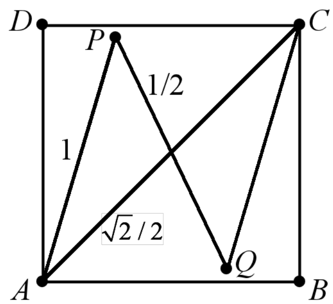
Answer: 308

Solution: We seek consecutive even perfect squares $(2a-2)^2$, $(2a)^2$, and $(2a+2)^2$ such that the triangle inequalities holds. It suffices to find the smallest positive integer a such that $(2a-2)^2 + (2a)^2 > (2a+2)^2$, which simplifies to $a(a-4) > 0$, from which we see that $a = 5$. Hence, the triangle has side lengths 64, 100, and 144, and perimeter $64 + 100 + 144 = \boxed{308}$.

Problem 2 In square $ABCD$, an ant travels from A to the opposite vertex C along a zigzag path that starts at A, goes straight to P, then straight to Q, then straight to C, where $AP = PQ = QC = CD$ and $m\angle APQ = m\angle PQC$. What is $\cos(m\angle APQ)$? Express your answer as a fraction in simplest form.

Answer: $\dfrac{3}{4}$

Solution:



By symmetry, the path passes through the center of the square. Thus, the path and the diagonal $\overline{AC}$ form two congruent triangles each with sides of length 1, 1/2, and $\sqrt{2}/2$. Using the law of cosines, we find that

$$(\sqrt{2}/2)^2 = 1^2 + (1/2)^2 - 2(1)(1/2)\cos(m\angle APQ),$$

from which we find $\cos(m\angle APQ) = \boxed{3/4}$.

Problem 3 How many positive integers divide evenly into $19^8 - 2^{12}$?

Answer: 96

Solution: We compute the prime factorization:

$$\begin{aligned}
19^8 - 2^{12} &= (19^4 + 2^6)(19^4 - 2^6) \\
&= (19^4 + 76^2 + 2^6 - 76^2)(19^2 + 2^3)(19^2 - 2^3) \\
&= ((19^2 + 2^3)^2 - 76^2)(369)(353) \\
&= (19^2 + 2^3 + 76)(19^2 + 2^3 - 76)(9)(41)(353) \\
&= (445)(293)(3^2)(41)(353) \\
&= (3^2)(5)(41)(89)(293)(353).
\end{aligned}$$

Thus, $19^8 - 2^{12}$ has $3 \cdot 2^5 = \boxed{96}$ divisors.

Problem 4 You have a rectangular grid of squares, 3 columns wide and 6 rows high. How many ways are there to color half of its 18 squares black and the other half white so that no two rows have the exact same pattern of black and white squares?

Answer: 7200

Solution: If we think of each row as representing a binary number with black representing the digit 1 and white representing the digit 0, then the 6 rows give us 6 distinct numbers between 0 and 7, inclusive. The total number of times the binary digit 1 occurs in these 8 numbers is $8 \cdot 3/2 = 12$. Therefore, the two missing numbers must have a total of 3 digits that are 1, and, conversely, if we pick any two numbers that have a total of 3 digits that are 1, then the other 6 will have 9 binary digits equal to 1. There are 10 pairs of numbers from $\{0, 1, 2, 3, 4, 5, 6, 7\}$ such that the two numbers have a total of 3 binary digits that are 1: $\{0, 7\}$, $\{1, 3\}$, $\{1, 5\}$, $\{1, 6\}$, $\{2, 3\}$, $\{2, 5\}$, $\{2, 6\}$, $\{4, 3\}$, $\{4, 5\}$, and $\{4, 6\}$. Therefore, there are 10 sets of 6 numbers from $\{0, 1, 2, 3, 4, 5, 6, 7\}$ which could serve as the set of numbers we find in the rows of our grid. These numbers can be placed into the rows in any order. Thus, there are $10(6!) = \boxed{7200}$ possible such colored grids.

Problem 5 The graph of a cubic polynomial has a local maximum at $(-10, 10)$ and a local minimum at $(10, -10)$. What is its leading coefficient? Express your answer as a fraction in simplest form.

Answer: $\dfrac{1}{200}$

Solution: Let $p(x)$ be the polynomial. Then for some constants A, r, B, and s, we have $p(x) - 10 = A(x+10)^2(x-r)$ and $p(x) + 10 = B(x-10)^2(x-s)$. Thus, $10 + A(x+10)^2(x-r) = -10 + B(x-10)^2(x-s)$. Comparing coefficients, we find that

$$\begin{aligned} A &= B \\ A(20-r) &= -B(20+s) \\ A(100-20r) &= B(20s+100) \\ 10 - 100Ar &= -10 - 100Bs \end{aligned}$$

The first and third equations tell us that $r = -s$. The second equation then gives us $r = -s = 20$. The fourth equation can then be written $10 - 2000A = -10 + 2000A$, from which we deduce that the lead coefficient is $A = \boxed{1/200}$.

Problem 6 Let C be a circle with radius 1 and center O. Let A and B be the endpoints of a 90° arc on the circumference of C. The circumference of a circle D, with center Q, intersects the circumference of C at points A and B in right angles. Let P be one of the points on the circumference of D such that $\overline{OP}$ forms a 60° angle with the tangent line to D at P. Determine $\cot^2(m\angle POQ)$.

Answer: 7

Solution: Let $X = m\angle POQ$. Note that circle D has radius 1 and $OQ = \sqrt{2}$. Also, $m\angle OPQ = 30^\circ$. Applying the law of sines to $\triangle POQ$, we get $1/\sin X = \sqrt{2}/\sin 30^\circ = \sqrt{8}$. Hence,

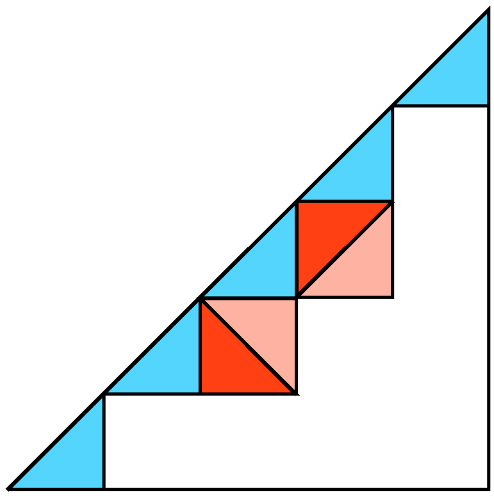
$$\cot x = \cos x/\sin x = \sqrt{1 - 1/8} \cdot \sqrt{8} = \sqrt{7},$$

and the answer is $\boxed{7}$.

Problem 7 In how many ways can an isosceles right triangle with legs of length 5 be tiled by isosceles right triangles with legs of length 1?

Answer: 1024

Solution:



Comparing areas, we find that there are 25 tiles in a tiling. Because $\sqrt{2}$ is irrational, if a, b, c, and d are rational, then $a + b\sqrt{2} = c + d\sqrt{2}$ implies $a = c$ and $b = d$. Therefore, the hypotenuse of the tiling area must be lined with 5 hypotenuses of tiles. Removal of these 5 tiles on the hypotenuse leaves a shape consisting of 10 unit squares in a staircase shape. We claim that these squares must each be filled with a pair of tiles joined along their hypotenuses. To see this, note that a unit length side along the stairs must be attached to the leg of a tile, and the hypotenuse of that tile must be attached to the hypotenuse of another tile. By filling in the 4 squares abutting the stairs, there results a smaller staircase consisting of 6 unit squares, which we continue filling with tiles in a similar manner. Since there are 2 ways to place each pair of tiles into a unit square, there are a total of $2^{10} = \boxed{1024}$ tilings.

Problem 8 Let C be a circle. Let S be the set of points in C that are the centroids of obtuse triangles inscribed in C. What fraction of the area of C is occupied by S? Express your answer as a fraction in simplest form. (The centroid of a triangle, also known as the center of mass, is the point where the three medians intersect.)

Answer: $\dfrac{8}{9}$

Solution: Let U be the set of points in the interior of C and outside the circle concentric with C and of radius $1/3$ that of C. We claim that $U = S$.

First, we show that $U \subseteq S$. Let P be in U. Let d be the diameter of C that contains P. Let A be the endpoint of d on the same side of the midpoint of d as P. Let Q be the point on d not between P and A such that $QP : QA = 1/3$. Since P is in U, Q is also on the same side of the midpoint of d as A. Let BC be the chord of C that is perpendicular to d and intersects d at Q. Then P is the centroid of ABC, and ABC is obtuse (since $m\angle BAC > 90°$). Therefore, P is in S.

Next, we show that $S \subseteq U$. Let T be any obtuse triangle inscribed in C. Label the vertices of T with A, B, and C so that BC is the side opposite the obtuse angle. Place a coordinate system with origin at the center of C such that BC is parallel to the vertical axis and intersects the horizontal axis at $(d, 0)$ with $d > 0$. Let r be the radius of C. Since A is separated from the origin by $\overleftrightarrow{BC}$, we know that the coordinates of A are given by $r(\cos\theta, \sin\theta)$, where θ is such that $r\cos\theta > d$. Thus, the centroid of $\triangle ABC$ is located at $(d, 0) + (1/3)(r\cos\theta - d, r\sin\theta)$. The square of the distance of this point from the origin is

$$(\frac{2d + r\cos\theta)}{3})^2 + (\frac{r\sin\theta}{3})^2 = \frac{4d^2 + 4rd\cos\theta + r^2}{9} > \frac{8d^2 + r^2}{9} > \frac{r^2}{9}.$$

Hence the distance of the centroid of T from the origin exceeds $r/3$ and $S \subseteq U$.

Thus, $U = S$ and the fraction of the area of C occupied by S is $1-(1/3)^2 = \boxed{8/9}$.

Problem 9 Six points are chosen uniformly and independently at random on the boundary of an equilateral triangle with perimeter P. ("Uniformly" means that the probability that a point lies on a particular segment of the boundary of length L is equal to L/P.) What is the probability that the convex hull of the 6 points is a quadrilateral? Express your answer as a fraction in simplest form.

Answer: $\frac{80}{243}$

Solution: Favorable outcomes are the following: (1) 4 points on one side and 2 on another, (2) 4 points on one side and 1 on each of the other sides, or (3) 3 points on each of two sides. (We can ignore situations where a vertex is chosen because that is a zero-probability event.) If we label the sides of the triangle by A, B, and C, we can think of the various outcomes as a string of 6 such letters, one for each chosen point. Each string occurs with probability $1/3^6$.

For outcome (1), there are 3 ways to choose a pair of letters and 2 ways to select the letter to appear 4 times, and there are $\binom{6}{4}$ different ways to arrange these letters in the string. Thus, the probability of outcome (1) is $3 \cdot 2 \cdot \binom{6}{4}(1/3)^6 = 90/3^6$.

For outcome (2), there are 3 choices for the letter that appears 4 times and no choice but to use the remaining 2 letters each once, and there are $2\binom{6}{4}$ ways to arrange these letters in the string. Thus, the probability of outcome (2) is $3(2\binom{6}{4})(1/3)^6 = 90/3^6$.

For outcome (3), there are 3 ways to choose a pair of letters, and there are $\binom{6}{3}$ ways to arrange these letters in the string. Thus, the probability of outcome (3) is $3\binom{6}{3}(1/3)^6 = 60/3^6$.

Adding these probabilities, we find that the answer is $(90+90+60)/3^6 = \boxed{80/243}$.

Problem 10 Let F_n be the Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for integer $n > 1$. How many integers have the form $F_a - F_b$ where $1 \le b < a \le 30$?

Answer: 380

Solution: Since $F_1 = F_2$, the answer does not change if we restrict to $1 < b < a \le 30$ (except that we have to be sure to account for $0 = F_2 - F_1$ in our final tally!). There are $\binom{29}{2} = 406$ pairs (a, b) that satisfy the inequalities. We determine which pairs correspond to the same difference of Fibonacci numbers. Suppose $F_a - F_b = F_c - F_d$, where $a > b$ and $c > d$. Note that $F_a - F_b \ge F_{a-2}$ and $F_c - F_d \ge F_{c-2}$. Therefore, $a < c+2$ and $c < a+2$, that is, $a = c-1$, c, or $c+1$. If $a = c$, then $b = d$ and $(a, b) = (c, d)$. If $a = c+1$, then $F_a = F_c + F_{c-1}$, so $F_{c-1} + F_d = F_b$, which shows that $d = c-2$ and $b = c$. That is $(a, a-1)$ produces the same difference as $(a-1, a-3)$. If $a = c-1$, then $F_c = F_a + F_{a-1}$ so that $F_d = F_b + F_{a-1}$, which shows that $b = a-2$ and $d = a$, which again shows that $(a, a-2)$ produces the same results as $(a+1, a)$. Therefore, the differences arising from the 406 pairs include each of the Fibonacci numbers F_x, for values of x from 2 to 28, twice. Hence, the final answer is $1 + (406 - 27) = \boxed{380}$.

Problem 11 The quartic polynomial p has integer coefficients and 4 distinct positive integer roots. If $p(4) = -256$ and $p(5) = -135$, what are its roots? Write your answer as a list of numbers in increasing order separated by commas.

Answer: $2, 6, 8, 20$

Solution: We know $p(x) = q(x-a)(x-b)(x-c)(x-d)$ for some integer q and distinct positive integers a, b, c, and d.

We also know that $p(4) = q(4-a)(4-b)(4-c)(4-d) = -256$, which shows that for any root r of $p(x)$, the number $4-r$ must be a factor (possibly negative) of 256. Similarly, the number $5-r$ must be a factor of 135. That is, r must be in both sets

$$\{-252, -124, -60, -28, -12, -4, 0, 2, 3, 5, 6, 8, 12, 20, 36, 68, 132, 260\}$$

and

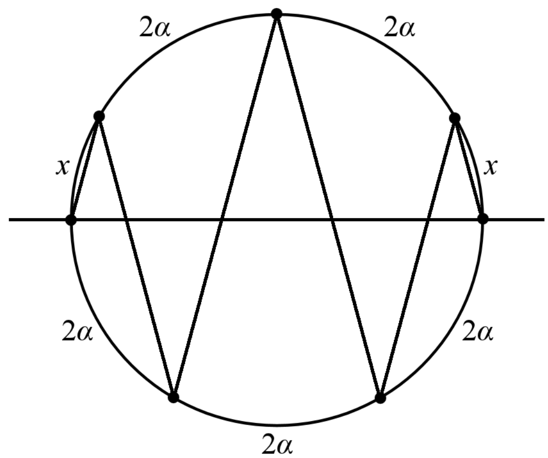
$$\{-130, -40, -22, -10, -4, 0, 2, 4, 6, 8, 10, 14, 20, 32, 50, 140\}.$$

The only common positive integers in these two sets are 2, 6, 8, and 20, which must therefore be the roots of $p(x)$, hence, the answer is $\boxed{2, 6, 8, 20}$.

Problem 12 A circle of radius r is drawn in the xy-coordinate plane centered at the point $(r, 0)$. An ant walks a zigzag path that begins at the origin with slope m. Each time the ant meets either the circumference of the circle or the x-axis, the ant changes direction by negating the slope of the line it is on. The ant always drifts to the right. The ant reaches $(2r, 0)$ after creating a zigzag path with 10 segments. If the first segment of the ant's journey is 1 unit long, what is the total distance that the ant traveled? Write your answer in the form $a + b\sqrt{c}$, where a, b, and c are positive integers and c is square-free.

Answer: $8 + 4\sqrt{3}$

Solution:



The ant creates 5 similar isosceles triangles with bases on the x-axis. Let α and β be the measures of the apex and base angles, respectively, of these isosceles triangles. If the even numbered isosceles triangles (in the order created by the ant) are reflected over the x-axis, we get a pattern of inscribed angles that cut off arcs with measures as indicated in the figure. (In the figure, x is the measure of the arc subtended by the first leg of the ant's journey.) Now, $x = 180° - 2\beta = \alpha$. Thus, the ant meets the circle at a subset of the vertices of a regular dodecagon and $\beta = 75°$. Let L be the total distance traveled by the ant and let d be the diameter of the circle. Then $L\cos(75°) = d$. But also, $\cos(75°) = 1/d$. Thus, $L = 1/\cos^2(75°) = \boxed{8 + 4\sqrt{3}}$.

Problem 13 Let p be the unique polynomial of degree 6 such that

$$p(n) = (-1)^n \binom{6}{n}$$

for $n = 0, 1, 2, 3, \ldots, 6$. What is $p(7)$?

Answer: 1716

Solution: Since the polynomial has degree 6, the seventh difference operator of the sequence $p(n)$ is 0. That is, let $p_n = p(n)$ for integers n and define the operator S by $Sp_n = p_{n+1}$. Then $(S-1)^7 p_n = 0$. That is

$$(S^7 - \binom{7}{1}S^6 + \binom{7}{2}S^5 - \binom{7}{3}S^4 + \cdots + \binom{7}{6}S^1 - \binom{7}{7}S^0)p_0 = 0.$$

Thus,

$$p_7 - \binom{7}{1}\binom{6}{6} - \binom{7}{2}\binom{6}{5} - \binom{7}{3}\binom{6}{4} - \cdots - \binom{7}{6}\binom{6}{1} - \binom{7}{7}\binom{6}{0} = 0,$$

which simplifies to $p_7 - \binom{13}{7} = 0$ or $p_7 = \binom{13}{7} = \boxed{1716}$.

Problem 14 The set S contains 2024 elements. What is the size of a largest collection C of subsets of S with the property that the intersection of every three subsets in C is nonempty, but the intersection of every four is empty?

Answer: 24

Solution: Suppose there are n subsets in C. For any three distinct elements X, Y, and Z in C, the intersection $X \cap Y \cap Z$ is nonempty and must be disjoint from every other element of C (otherwise, there would be 4 elements of C whose intersection is nonempty contrary to hypothesis). There are $\binom{n}{3}$ such triples, so S must have at least $\binom{n}{3}$ elements. Since $2024 = \binom{24}{3}$, we must have $n \leq 24$.

We show that $n = 24$ by constructing an example. Let P consist of 24 generic planes in three-dimensional space. Here, "generic" means that every pair of distinct planes intersect in a line, every triple of distinct planes intersect in a single point, and any quadruple of distinct planes have empty intersection. (This can be done, for example, by picking 24 points in the upper hemisphere of a sphere so that no 3 lie on a great circle and no 4 lie on a circle, then taking the planes to be the tangent planes to the sphere at those 24 points.) Let S be the set of points that are the intersection of any 3 distinct planes in P. Let C consists of the 24 subsets of S obtained by taking the intersection of each plane in P with S. This explicit construction shows that the answer is $\boxed{24}$.

Problem 15 Let $\triangle ABC$ be equilateral with side length 999. Point P is in the interior of $\overline{BC}$. The distances from P to A, B, and C are all integers. What is AP?

Answer: 889

Solution: Let $d = AP$ and let $x = BP$. Let $T = m\angle APB$. Using the law of cosines twice we get

$$999^2 = d^2 + x^2 - 2xd\cos T$$

$$999^2 = d^2 + (999 - x)^2 + 2(999 - x)d \cos T.$$

Multiply the first equation by $(999 - x)$ and the second by x and add the results to get:

$$(999 - x)999^2 + x999^2 = (999 - x)(d^2 + x^2) + x(d^2 + (999 - x)^2),$$

which simplifies to:

$$x^2 - 999x + 999^2 - d^2 = 0.$$

For this to have an integer solution in x, the discriminant must be a perfect square, that is

$$999^2 - 4(999^2 - d^2) = 4d^2 - 3(999^2)$$

must be a perfect square. Call this square m^2. Then $(2d + m)(2d - m) = 3(999^2)$. If $0 < B < A$ is such that $3(999^2) = AB$, then we get the solution $d = (A + B)/4$ and $m = (A - B)/2$. We also need $999\sqrt{3}/2 < d < 999$, so $865 < (A + B)/4 < 999$. The only such solution occurs when $A = 3^7 = 2187$ and $B = 37^2 = 1369$, giving $d = (2187 + 1369)/4 = 3556/4 = \boxed{889}$.

Problem 16 For every positive integer n, let d_n be the greatest common divisor of $n^2 + 1$ and $n^2 + n + 10$. Let m be the maximum value attained by the sequence d_n. Let k be the smallest positive integer such that $d_k = m$. What is k?

Answer: 73

Solution: Applying the Euclidean algorithm, we find that

$$82 = (n - 8)(n^2 + 1) - (n - 9)(n^2 + n + 10).$$

Therefore d_n divides evenly into 82 for all positive integers n.

If we reduce modulo 41, our equation becomes

$$(n - 8)(n^2 + 1) = (n - 9)(n^2 + n + 10).$$

This implies that $n^2 + 1$ and $n^2 + n + 10$ have a common root, modulo 41, and this common root must be the root of $n^2 + 1$ other than 9, and hence must be $-9 = 32 \pmod{41}$.

Now n^2+1 is even when n is odd and n^2+n+10 is always even. Therefore $2 \mid d_n$ if and only if n is odd.

Thus, $d_n = 82$ if and only if $n = 32 \pmod{41}$ and n is odd. The smallest positive integer n that satisfies both of these conditions is $41 + 32 = \boxed{73}$.

Problem 17 Define $f_n(x)$ recursively by setting $f_0(x) = |x|$ and letting

$$f_n(x) = |n - f_{n-1}(x)|$$

for every integer $n > 0$. Let c be the unique positive integer such that $f_{100}(c) = 0$. What is the area sandwiched between the x-axis and the graph of $y = f_{100}(x)$ for $-c < x < c$?

Answer: 507500

Solution: Let $c_n = n(n+1)/2$ be the nth triangular number.

For positive integers n, let P_n denote the following multipart claim: The function f_n is even, piecewise linear, and has a unique positive real zero at c_n, and starting at $(c_n, 0)$ and moving to the left, these linear segments have slopes that alternate -1, $+1$, -1, $+1$, and have horizontal intervals of length n, $n-1$, $n-2$, …, 2, 1, respectively. Finally, for $|x| > c_n$, we have $f_n(x) = |x| - c_n$.

We shall prove P_n by induction on n.

The base case $n = 1$ can be checked directly.

Assume the claim P_n is true for some positive integer $n \geq 1$. First note that $f_{n+1}(-x) = |n+1-f_n(-x)| = |n+1-f_n(x)| = f_{n+1}(x)$, so f_{n+1} is an even function. Note that the description of the graph of f_n implies that $0 < f_n(x) < n+1$ for $-(c_n+n+1) < x < c_n+n+1$ and so the line $y = n+1$ intersects $y = f_n(x)$ in exactly two places: $(c_n+n+1, n+1)$ and $(-(c_n+n+1), n+1)$. Therefore $f_{n+1}(x)$ has two zeroes located at $\pm c_{n+1}$. Also, for $x \in [-(c_n+n+1), c_n+n+1]$, we have $f_{n+1}(x) = (n+1)-f_n(x)$. Since vertical translation does not affect the slope of a line but negation negates the slope, the description of the slopes of the piecewise linear segments of f_{n+1} are as described by P_{n+1} in the interval $[-(c_n+n+1), c_n+n+1]$. Outside of this interval, we know $f_n(x) = |x| - c_n$, so that $f_{n+1}(x) = |n+1+c_n-|x|| = |x| - (c_n+n+1)$, as desired.

Thus, P_n is true for all positive integers n.

Let A_n be the area between the x-axis and the graph of $y = f_n(x)$ for $-c_n < x < c_n$. If we draw in the line $y = n$, we see that $A_n + A_{n-1} + n^2 = 2nc_n$. That is, $A_n + A_{n-1} = n^3$. Since $A_{n-1} + A_{n-2} = (n-1)^3$, we find that $A_n - A_{n-2} = n^3 - (n-1)^3$. (For convenience, we extend A_n to $n = 0$ by defining $A_0 = 0$.) Telescoping, we find for even n,

$$A_n = \sum_{k=1}^{n/2} (2k)^3 - \sum_{k=1}^{n/2} (2k-1)^3,$$

which simplifies to $A_n = n^2(2n+3)/4$. (Recall that $\sum_{k=1}^{n} k^3 = (n(n+1)/2)^2$.) Thus, $A_{100} = 100^2(203)/4 = \boxed{507500}$.

Problem 18 Consider the 24 points in four-dimensional space whose coordinates are the various permutations of 1, 2, 3, and 4. Among these 24 points, how many subsets of 4 form the vertices of a square?

Answer: 18

Solution: Throughout, we indicate the point (a, b, c, d) with $abcd$. Also note that permutation of the coordinates is on orthogonal transformation. We exploit these symmetries to help us count.

But first, we take a moment to create the following convenient chart showing the square of the distance of each of the 24 points from 1234:

Points	Square of distance to 1234
1234	0
1243, 1324, 2134	2
2143	4
1342, 1423, 2314, 3124	6
1432, 3214	8
2413, 3142	10
2341, 4123	12
2431, 3241, 4132, 4213, 4312	14
3412	16
3421, 4231	18
4321	20

In a square, the diagonal length squared is twice the square of its side length. From our chart, we deduce that there are no squares with squared side length greater than 10. We now split into cases defined by the possible side lengths s of the square. For each case, we find the squares that have 1234 as a vertex, then determine the total number of squares for that case.

If $s^2 = 10$, our convenient chart shows that only the vertices 2413, 3142, and 4321 could possibly form a square with 1234. Computing distances between pairs of these 4 points confirms that they are the vertices of a square. Thus, each point belongs to 1 such square, and since there are 4 vertices per square, we get a total of 6 squares.

If $s^2 = 8$, our convenient chart shows that only the vertices 1432, 3214, and 3412 could possibly form a square with 1234. Computing distances between pairs of these 4 points confirms that they are the vertices of a square. Thus, each point again belongs to 1 such square, and we again get 6 such squares total.

If $s^2 = 6$, the only possible vertices connected to 1234 by an edge of the square, according to our convenient chart, are: 1342, 1423, 2314, and 3124. The relative displacement vectors from 1234 to these 4 points are $(0, 1, 1, -2)$, $(0, 2, -1, -1)$, $(1, 1, -2, 0)$, and $(2, -1, -1, 0)$, respectively. A quick check of dot products reveals that no pair of these displacements are perpendicular, hence, there is no square with $s^2 = 6$.

If $s^2 = 4$, our convenient chart shows that there is no such square because there is only 1 point of distance 2 from 1234.

Finally, if $s^2 = 2$, our convenient chart tells us that the only such square involving 1234 must have 2143 across the diagonal from 1234. Of the points 1243, 1324, and 2134, only 1243 and 2134 are $\sqrt{2}$ away from both 1234 and 2143. Since the distance between 2134 and 1243 is 2, the 4 points 1234, 2134, 1243, and 2143 are, indeed, the vertices of a square. We again get a total of 6 such squares.

In total, there are $\boxed{18}$ squares.

Problem 19 The sequence a_n is defined recursively as follows: $a_1 = \tan(15°/2)$ and

$$a_{n+1} = \frac{1 - 2a_n - a_n^2}{1 + 2a_n - a_n^2},$$

for every integer $n \geq 1$. Determine the number A such that $-90 < A < 90$ and $a_{100} = \tan A°$.

Answer: 75

Solution: Observe that

$$\frac{1 - 2\tan x - \tan^2 x}{1 + 2\tan x - \tan^2 x} = \frac{1 - \frac{2\tan x}{1 - \tan^2 x}}{1 + \frac{2\tan x}{1 - \tan^2 x}} = \frac{1 - \tan(2x)}{1 + \tan(2x)} = \tan(45° - 2x).$$

That is, if $a_n = \tan x$, then $a_{n+1} = \tan(45° - 2x)$.

From $a_1 = \tan(15°/2)$, we compute the next few terms: $a_2 = \tan(30°)$, $a_3 = \tan(-15°)$, $a_4 = \tan(75°)$, $a_5 = \tan(-105°) = \tan(75°)$. Thus, we must have $a_n = \tan(75°)$ for all $n \geq 4$ and $A = \boxed{75}$.

Problem 20 Let ABC be a triangle with $\angle ACB$ right. Let M be the midpoint of the hypotenuse. Let O be the foot of the altitude from C. Let N be the intersection of the angle bisector at C and the hypotenuse. Given that $MN = 53$ and $NO = 45$, determine the length of the hypotenuse.

Answer: 371

Solution: We shall solve a scaled version of the problem where $MN = 1$ and $ON = x$. Using standard labeling, we let $a = BC$, $b = CA$, and $c = AB$.

By relabeling A and B if necessary, we may assume that $a < b$. (If $a = b$, then $M = N = O$, so $a \neq b$.)

Note that $AM = c/2$ and, by the angle bisector theorem, $AN = bc/(a+b)$. Since we have scaled so that $MN = 1$, we have

$$\frac{bc}{a+b} - \frac{c}{2} = 1.$$

Rearranging, this becomes

$$\frac{c}{2} = \frac{b+a}{b-a},$$

which is equivalent to

$$\frac{c}{2} = \tan(A + 45^\circ),$$

or $A = \arctan(c/2) - 45^\circ$.

Since $NO = x$, we have

$$x = b \cos A - (\frac{c}{2} + 1).$$

Substituting $c \cos A$ for b, we get $x = c \cos^2(A) - (c/2 + 1)$. Substituting $\arctan(c/2) - 45^\circ$ for A, we obtain an equation which we can solve for c in terms of x. Doing so, we find that

$$c^2 = 4\frac{1+x}{1-x}.$$

Setting $x = 45/53$, we find $c = 7$. Thus, the original hypotenuse is $53 \cdot 7 = \boxed{371}$.