Solutions to Math Prize Olympiad 2025

Advantage Testing Foundation / Jane Street

Fall/Winter 2025

§1 Problem 1, proposed by Oleg Kryzhanovsky

A positive integer is called *amazing* if you can erase one of its digits (in base 10), optionally reorder the remaining digits, multiply the result by 9, and get the original number. For instance, the number 315 is amazing because $35 \cdot 9 = 315$. Determine whether there are infinitely many amazing numbers not divisible by 10.

There are indeed infinitely many such numbers. We claim

$$N = 2025\,2025\,2025\,2025\dots2025$$

is amazing (for any number of repetitions of 2025). Indeed, note that

$$\frac{N}{9} = 225\,0225\,0225\,0225\dots0225.$$

which is indeed a rearrangement of the digits of N after one 0 is deleted.

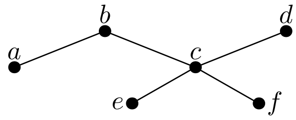
> **Remark.** The idea of the solution is based on the number $M = 2025$, which has the property that one can delete 0 to get $M/9$. This solution can be made to work with other such numbers like $M = 405 = 9 \cdot 45$. One way to find 405 is to just solve $100x + z = 9(10x + z)$ over integers.
>
> It turns out there are only finitely many integers M not divisible by 10 for which deleting a single digit gives $M/9$.

> **Remark.** The above construction has the cute property it uses the current year, but it is far from the only construction. Other examples of possible constructions are
>
> $$9 \cdot 9000\cdots001 = 81000\cdots0009$$
>
> and
>
> $$9 \cdot 450450\cdots045 = 405405\cdots405.$$

§2 Problem 2, proposed by Carlos Rodriguez

Let $n \geq 2$ be an even integer and let T be a tree with n vertices. Show that T has an edge between two vertices whose degrees are either both even or both odd.

(Here, a *tree* is defined as a set of n *vertices* and $n-1$ *edges* between pairs of vertices, such that any two vertices are linked via a path of one or more edges. The *degree* of a vertex is defined as the number of edges using that vertex. For example, if $n = 6$ and T is a tree with 5 edges ab, bc, cd, ce, and cf as illustrated below, then edge bc joins two even-degree vertices: b has degree 2 while c has degree 4.)



We prove the following contrapositive statement instead:

Proposition

Let T be a tree where every edge joins an odd-degree vertex to an even-degree vertex. Then n is odd.

¶ First solution (double-counting, from author). We actually prove the following, which shows that the "tree" hypothesis is unnecessarily strong:

Claim — In *any* finite graph for which edges join odd-degree to even-degree vertices, the number of edges is even.

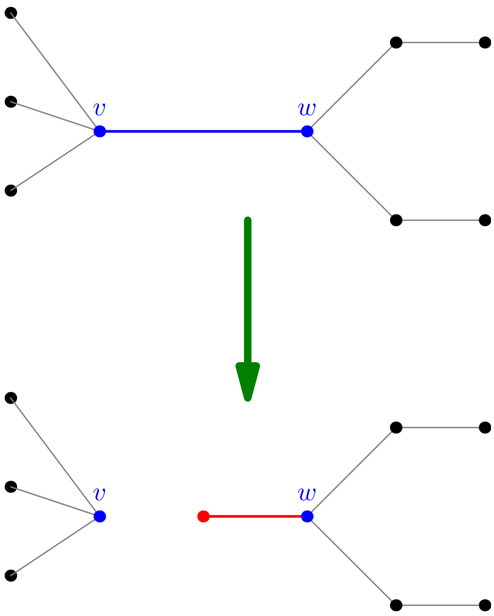
Proof. Each edge touches exactly one even-degree vertex. Hence the number of edges equals the sum of the degrees of the even-degree vertices. □

In a tree, the number of edges is $n - 1$, so n is odd.

¶ Second solution (induction, Evan Chen). We prove the above proposition by induction on n with the cases $n = 1$ being vacuous.

First, suppose that all odd-vertex degrees are degree 1. Then it is easy to see that the graph must be a star, i.e. it consists of a single vertex v connected to several other vertices of degree 1. In particular, $\deg v$ is even. Then $n = \deg v + 1$ is odd.

Otherwise, we let w be an odd-degree vertex with $\deg w \geq 3$. Then let v be a neighbor of w, so that $\deg v \geq 2$ is even. We consider deleting the edge vw, then adding a new vertex of degree 1 just to w. This surgery splits T into two new trees each with at least 2 vertices. An example of this surgery is shown in the figure below.



Since there are $n+1$ total vertices now, it follows each of the two new trees has at most $n-1$ vertices. The induction hypothesis applies now, and both new trees have an odd number of vertices. Hence $n+1$ is even, so n is odd, completing the induction.

§3 Problem 3, proposed by Luke Robitaille

Jessica and Hannah play a game. First, Jessica draws a circle Γ and chooses a set S of 13 points on Γ. Hannah sees Jessica's choices and then chooses a set T of 2025 points on Γ. Jessica sees what Hannah chooses.

Then Jessica performs a sequence of operations that modify the set S. In an operation, Jessica chooses two perpendicular chords AB and PQ of Γ, for which A, B, P are all in S, but Q is not. She then removes A, B, and P from S and adds Q into S.

Jessica wins if she reaches a state where S has only one point, and that point is not in T. Otherwise, Hannah wins. Which player has a winning strategy?

The answer is that Hannah wins.

We consider Γ as the unit circle in the complex plane henceforth. In the complex plane, the operation can be rephrased as

$$(a, b, p) \longmapsto q = -\frac{ab}{p}.$$

The main point of the problem is the following is true:

Claim — If Jessica performs a sequence of operations combining some $2k+1$ points $z_1, \ldots, z_{2k+1}$ of S into one point, then the final point has coordinates of the form

$$(-1)^k \cdot \frac{\text{the product of some } k+1 \text{ of the } z_i}{\text{the product of the other } k \text{ of the } z_i}.$$

Proof. This follows directly by induction on k (with the base case $k \le 1$ following from the comment above). To write out the (admittedly hideous) notation for the inductive step, assume without loss of generality that Jessica's first move takes z_1, z_2, z_3 and replaces them with $w = -\frac{z_1 z_2}{z_3}$.

Now we are left with $\{w, z_4, z_5, \ldots, z_{2k+1}\}$. The induction hypothesis (together with permuting the indices 4 through $2k+1$, by symmetry) symmetry means we may assume the final point is either

$$(-1)^k \frac{w z_4 \ldots z_{k+3}}{z_{k+4} \ldots z_{2k+3}} = (-1)^{k+1} \frac{z_1 z_2 \cdot z_4 \ldots z_{k+3}}{z_3 \cdot z_{k+4} \cdot z_{2k+3}}$$

or

$$(-1)^k \frac{z_4 \ldots z_{k+4}}{w z_{k+5} \ldots z_{2k+3}} = (-1)^{k+1} \frac{z_3 \cdot z_4 \ldots z_{k+4}}{z_1 z_2 \cdot z_{k+5} \ldots z_{2k+3}},$$

both of which have the desired form. □

Returning to the main problem, we see that the number of possibilities for Jessica's point is at most

$$\binom{13}{6} = 1716 < 2025$$

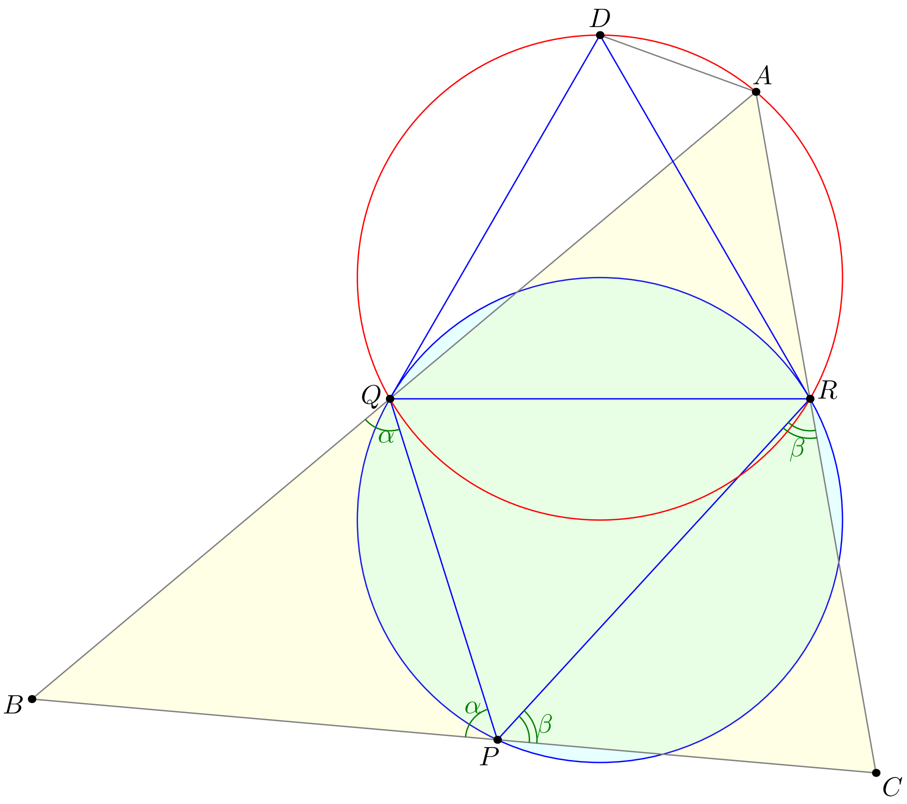
since the expression in the claim is determined by the choice of which 6 of the z_i's are in the denominator and which are in the numerator. Hence Hannah wins.

§4 Problem 4, proposed by Karthik Vedula

Acute triangle PQR is inscribed in triangle ABC, with points P, Q, R lying in the interiors of sides BC, AB, AC, respectively, such that $BP = BQ$ and $CP = CR$. Suppose the tangents to the circumcircle of triangle PQR at Q and R meet at point D. Prove that

$$DB + DC \geq AB + AC.$$

In what follows, we use directed angles modulo 180 and abbreviate $P := \measuredangle RPQ$, $Q := \measuredangle PQR$, $R := \measuredangle QRP$, so $P + Q + R = 0$.



We start with the following claim:

Claim — The points D, A, Q, R are concyclic.

Proof. The basic idea is to calculate $\angle QAR$ in terms of the angles involving $\triangle PQR$. For convenience, let us define $\alpha := \measuredangle BQP = \measuredangle QPB$ (as $BP = BQ$) and $\beta := \measuredangle CPR = \measuredangle PRC$ (as $CP = CR$), so $\alpha + P + \beta = 0$ (since P lies on line BC). In that case,

$$\begin{aligned}\measuredangle RQA + \measuredangle ARQ &= -\measuredangle BQR - \measuredangle PRC \\ &= -(\alpha + Q) - (\beta + R) \\ &= -(\alpha + \beta) - (Q + R) \\ &= 2P.\end{aligned}$$

Hence, $\measuredangle QAR = -2P$. Since $\measuredangle RQD = \measuredangle DRQ = P$, we also have $\measuredangle QDR = -2P$. This completes the proof. □

Claim — Ray DA is the external bisector of $\angle BAC$.

Proof. Note that D is the arc midpoint of the major arc of QR of the circumcircle of $DARQ$ (and this major arc contains A). □

Finally, let C' be the reflection of C over this external angle bisector, so that C' lies on line AB. Then the problem follows from:

$$DB + DC = BD + DC' \geq BC' = BA + AC' = AB + AC.$$

Remark. The final reflection step is not a new idea. It can be phrased more generally as saying that if ABC is a triangle, and ℓ is the external bisector of $\angle A$, then for any point D on ℓ we have $DB + DC \geq AB + AC$.

Another equivalent phrasing is that if B and C are fixed points and ℓ is a line not passing through them, then the point A minimizing $AB + AC$ is the one for which ℓ makes the same angle to lines AB and AC. This is the famous Heron's shortest path problem (sometimes phrased with a river instead of a mirror) and the construction of C' is the standard solution to this problem.

An even fancier rephrasing is to say that ℓ is the tangent to the ellipse with foci B and C passing through A. However, the use of conic sections seems to be cosmetic only.

Remark. If I is the center of (PQR), it's in fact not hard to see that I lies on $(DARQ)$ as well. Moreover, since $BPIQ$ and $CPIR$ are kites, it follows I is in fact the incenter of triangle ABC. The author's initial solution proved this as an intermediate step, but the solution above shows in fact I is not needed at all.