Math Prize for Girls
2025 Solutions

Problem 1 100 girls collected 2025 pounds of honey altogether. Each of the girls had an empty barrel. At one point they simultaneously did the following: each of the girls poured all her honey equally to all other girls, using their empty barrels. After that Katrine, one of the girls, had 20 pounds of honey. How much honey (in pounds) did Katrine have before honey redistribution?

Answer: 45

Solution: Let K be the number of pounds of honey that Katrine had before honey redistribution. Collectively, the other 99 girls have $2025 - K$ pounds of honey. Therefore, after redistribution, Katrine was given $(2025 - K)/99 = 20$ pounds of honey. Solving for K, we find $K = \boxed{45}$.

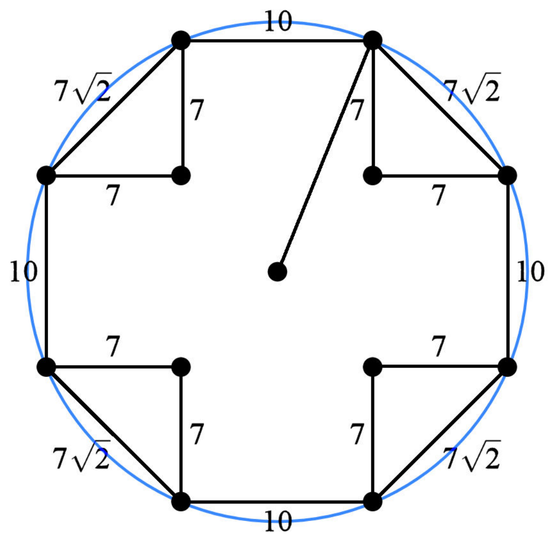
Problem 2 How many positive integers less than or equal to 1000 are there that are divisible by 2, 3, or 5, and if they are divisible by any two of the numbers 2, 3, or 5, then they are divisible by all three?

Answer: 501

Solution: Let $Z_k = \{n \in \mathbb{Z} \mid k \mid n \text{ and } 1 \le n \le 1000\}$. If we add up the sizes of Z_2, Z_3, and Z_5, we double count numbers that are divisible by exactly two of 2, 3, and 5, and triple count those that are divisible by all three. We do not want to count numbers divisible by exactly two of 2, 3, and 5, so we subtract $2(\#Z_2 \cap Z_3 + \#Z_3 \cap Z_5 + \#Z_5 \cap Z_2)$. However, after we've done that, we've counted numbers divisible by 2, 3, and 5 a total of -3 times, and since we want to account for such numbers, we must add back $4\#Z_2 \cap Z_3 \cap Z_5$. Thus, the answer is $500 + 333 + 200 - 2(166 + 66 + 100) + 4(33) = \boxed{501}$.

Problem 3 An octagon is inscribed in a circle. It has four consecutive sides of length 10 and four consecutive sides of length $7\sqrt{2}$. What is the radius of the circle?

Answer: 13



Solution: Note that the radius of the circumscribed circle does not change if we permute the sides of the octagon. If we permute the sides so that they alternate in length as you go around the perimeter, we obtain an octagon that is the convex hull of the red cross with 4 sides of length 10 and 8 sides of length 7. The radius of the circumscribed circle is therefore the hypotenuse of a right triangle with legs of lengths 5 and 12. Thus, the answer is $\boxed{13}$.

Problem 4 A regular hexagon has side length $6 + 2\sqrt{3}$. The hexagon is divided into 6 non-overlapping triangles by connecting each vertex to its opposite vertex. These triangles are then colored alternately black and white, going around the center. An identically colored congruent hexagon is placed over the first covering it precisely, then rotated about its center by 30° clockwise. What is the total area of the regions where black overlaps black?

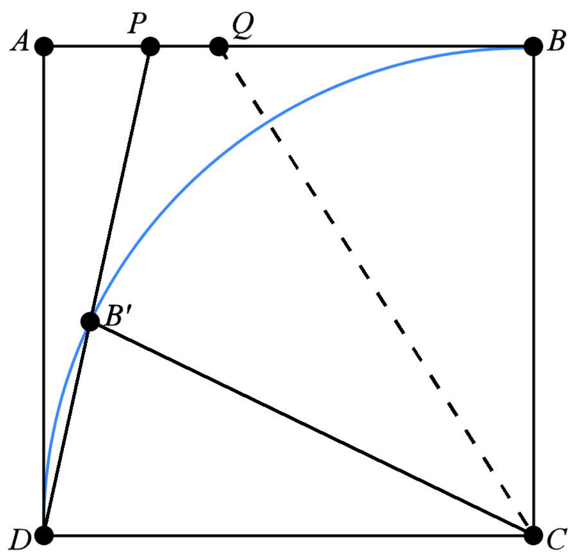
Answer: 54

Solution: Let $s = 6 + 2\sqrt{3}$. Note that the colored triangles are equilateral. Because the angular width of each triangle is 60°, the only way overlapping black regions can occur is via two black triangles related by a 30° rotation. There are 3 such regions, and each region is a kite composed of the union of two congruent right triangles with a legs of length $s\cos(30^\circ)$ and $s\cos(30^\circ)\tan(15^\circ)$. Therefore, the answer is

$$3s^2\cos^2(30^\circ)\tan(15^\circ) = 3s^2\cos^2(30^\circ)\frac{\sin(30^\circ)}{1+\cos(30^\circ)} = \boxed{54}.$$

Problem 5 Square $ABCD$ is a sheet of paper. Point P on the interior of side $\overline{AB}$ satisfies $m\angle ADP = 12°$. Point Q is on the interior of $\overline{PB}$. When the paper is folded along a crease that passes through C and Q, vertex B falls upon $\overline{DP}$. What is the degree measure of $\angle QCB$?

Answer: 33



Solution: Note that the locus of reflections of B in lines that pass through C form a circle centered at C, so the reflection of B must land at the intersection of this circle with $\overline{DP}$ other than D (because to reflect B to C, we must reflect in the diagonal of the square which would place Q at A). Let B' be this point of intersection. Because $\overline{DA}$ is tangent to the circle, we see that $m\angle B'CD = 2m\angle B'DA = 24°$. The crease $\overline{CQ}$ bisects $\angle BCB'$, which is complementary to $\angle B'CD$. Thus, the degree measure of $\angle QCB$ is $\boxed{33}$.

Problem 6 For every nonnegative integer n, let a_n be the number of integers m between 1 and 3^{2n}, inclusive, such that either m or $m/3$ is a perfect square. Let $b_n = a_{n+1} - 3a_n$. What is

$$\sum_{k=0}^{\infty} \frac{b_k}{3^k}\ ?$$

Express your answer as a radical in simplest form.

Answer: $\sqrt{3}$

Solution: The number of squares between 1 and 3^{2n}, inclusive, is 3^n. The number of numbers that are three times a perfect square in the same range is $\lfloor \sqrt{3} \cdot 3^{n-1} \rfloor$. Thus, a_n is equal to the sum of these two quantities, which is $\lfloor 3^{n-1}(3+\sqrt{3}) \rfloor$. Therefore, b_n is the base 3 digit of $\sqrt{3}$ in the 3^n's place. Hence, the answer is $\boxed{\sqrt{3}}$.

Problem 7 Let Q be the set of quartic polynomials with leading coefficient -1, four distinct positive integer roots less than or equal to 100, and whose graph has a vertical line of symmetry. What is the maximum value that any of these quartic polynomials can achieve over the real numbers?

Answer: 1,500,625

Solution: Let $q(x) \in Q$ and let $x = t$ be its vertical line of symmetry.

Then $q(x+t)$ has the y-axis as a line of symmetry which means that $q(x+t)$ is a quartic of the form $-x^4 + bx^2 + c$. Its roots must be symmetric about the y-axis so can be written as $-u$, $-v$, v, and u, with $0 < v < u$. (Note that we cannot have $v = 0$ since the roots of $q(x)$ are distinct.) Given these roots, we have $q(x+t) = -(x^2 - u^2)(x^2 - v^2)$. This expression is maximized when $x^2 = (u^2+v^2)/2$, with a maximum value of $((u^2-v^2)/2)^2$, which is maximized when $u - v$ is maximized, which occurs when the roots of $q(x)$ are 1, 50, 51, and 100. For these roots, the maximum is $((49.5^2 - 0.5^2)/2)^2 = \boxed{1{,}500{,}625}$.

Problem 8 Evaluate

$$\sum_{n=0}^{\infty} \frac{\cos(\pi n/6)}{\sqrt{3}^{\,n}}.$$

Express your answer as a fraction in simplest form.

Answer: $\dfrac{3}{2}$

Solution: Because cosine is periodic and the sum converges absolutely, we may separate the sum into 6 geometric series with different first terms, but the same common ratio of $-1/27$. Specifically, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\cos(\pi n/6)}{\sqrt{3}^{\,n}} &= \frac{\frac{1}{3^0} + \frac{\sqrt{3}/2}{3^{1/2}} + \frac{1/2}{3^1} + \frac{0}{3^{3/2}} - \frac{1/2}{3^2} - \frac{\sqrt{3}/2}{3^{5/2}}}{1 + \frac{1}{27}} \\
&= \frac{1 + \frac{1}{2} + \frac{1}{6} - \frac{1}{18} - \frac{1}{18}}{1 + \frac{1}{27}} \\
&= \frac{28/18}{28/27} \\
&= \boxed{\frac{3}{2}}.
\end{aligned}$$

Problem 9 A triangle has an inscribed circle of radius 3. One side of the triangle is split into segments of length 1 and 10 by the point where the inscribed circle touches the side. What is the perimeter of the triangle?

Answer: 220

Solution: Let x, 1, and 10 be the distances of each vertex of the triangle to the nearest point of tangency (of the circle with a side of the triangle). We have

$$3(x + 11) = \sqrt{(x + 11)x(10)(1)},$$

since the left side of the equation is the area of the triangle in terms of the semiperimeter and radius, and the right side is the area according to Heron's formula. Solving for x, we find $x = 99$. Therefore, the answer is $2(x + 11) = \boxed{220}$.

Problem 10 You want to make an edge-to-edge tiling of a 10×10 square, without gaps or overlaps, using tiles that are all similar to the 1×2 rectangle (but do not necessarily have integer side lengths). In an "edge-to-edge" tiling, no vertex of one tile lies in the interior of an edge of another. Given that the tile in the lower left corner is a vertical 1×2 rectangle, how many ways are there to do this?

Answer: 104

Solution: Note that the lowest row of tiles consists of tiles whose height is 2. The widths are 1 or 4. If any of these tiles has width 4, then the rest of the tiling must consist of a stack of rows identical to this bottom one. For there to be a tile with width 4 in the bottom row, there can be 2 of width 4 and 1 of width 1, or 1 of width 4 and 5 of width 1, for a total of $3 + 6 = 9$ ways.

If all tiles on the bottom row have width 1, then the tiling consists of replicating the first column across the tiling. The first column consists of tilings all of width 1 and of heights either 2 or 1/2, and any way to split into such tilings works. The stack atop the lower left tile can consist of 4 tilings of height 2 and 0 of height 1/2, 3 of height 2 and 4 of height 1/2, 2 of height 2 and 8 of height 1/2, 1 of height 2 and 12 of height 1/2, or 0 of height 2 and 16 of height 1/2, for a total of $\binom{4}{4}+\binom{7}{3}+\binom{10}{2}+\binom{13}{1}+\binom{16}{0} = 1+35+45+13+1 = 95$.

In total, there are $9 + 95 = \boxed{104}$ ways.

Problem 11 Solve for a:

$$\begin{aligned} a - b + c - d + e &= 1 \\ a - 2b + 4c - 8d + 16e &= 128 \\ a - 3b + 9c - 27d + 81e &= 2187 \\ a - 4b + 16c - 64d + 256e &= 16384 \\ a - 5b + 25c - 125d + 625e &= 78125. \end{aligned}$$

Answer: 16,800

Solution: The equations show that the polynomial $x^7+ex^4+dx^3+cx^2+bx+a$ has the roots -1, -2, -3, -4, and -5. If r and s are the other two roots, then, by Vieta's formulas, we have $-1 - 2 - 3 - 4 - 5 + r + s = 0$, and (computing the sum of products of pairs of roots) $85 - 15(r + s) + rs = 0$ (since the coefficients of x^6 and x^5 are both 0). Therefore, $rs = 140$ and a, the constant term, is $-(-1)(-2)(-3)(-4)(-5)(140) = \boxed{16{,}800}$.

Problem 12 What is the remainder when you divide $526^{12} - 494^{12}$ by the prime number 1019? (Note that $526 \times 494 \equiv -1 \pmod{1019}$.)

Answer: 532

Solution: Note that $526 + 494 = 1 \pmod{1019}$. This fact combined with the noted fact that $526 \times 494 = -1 \pmod{1019}$ means that the sequence $F_n \equiv 526^n - 494^n$ satisfies the Fibonacci recurrence relation, modulo 1019. Note that $F_0 = 0 \pmod{1019}$ and $F_1 = 32 \pmod{1019}$. Since the 12th Fibonacci number is 144, the answer is $144 \cdot 32 = 4608 = \boxed{532} \pmod{1019}$.

Problem 13 Let $-90 \le A \le 90$ satisfy the equation

$$\sin(A^\circ) = \frac{\dfrac{1}{2} + \displaystyle\sum_{k=0}^{44} \sin(2k^\circ)}{\displaystyle\sum_{k=0}^{44} \sin((2k+1)^\circ)}.$$

What is A?

Answer: 89

Solution: Ptolemy's theorem on cyclic quadrilaterals, expressed in terms of sines, says that

$$\sin(X)\sin(Y) + \sin(Z)\sin(W) = \sin(X+Z)\sin(X+W),$$

for all X, Y, Z, and W, such that $X+Y+Z+W = 180^\circ$. We add up the following instances of Ptolemy's theorem

$$\begin{array}{rcl} \sin(45^\circ)\sin(45^\circ) + \sin(44^\circ)\sin(46^\circ) & = & \sin(89^\circ)\sin(91^\circ) \\ -(\ \sin(44^\circ)\sin(46^\circ) + \sin(44^\circ)\sin(46^\circ)\ ) & = & -\sin(88^\circ)\sin(90^\circ) \\ \sin(44^\circ)\sin(46^\circ) + \sin(43^\circ)\sin(47^\circ) & = & \sin(87^\circ)\sin(91^\circ) \\ -(\ \sin(43^\circ)\sin(47^\circ) + \sin(43^\circ)\sin(47^\circ)\ ) & = & -\sin(86^\circ)\sin(90^\circ) \\ \sin(43^\circ)\sin(47^\circ) + \sin(42^\circ)\sin(48^\circ) & = & \sin(85^\circ)\sin(91^\circ) \\ & \vdots & \\ -(\ \sin(1^\circ)\sin(89^\circ) + \sin(1^\circ)\sin(89^\circ)\ ) & = & -\sin(2^\circ)\sin(90^\circ) \\ \sin(1^\circ)\sin(89^\circ) + \sin(0^\circ)\sin(90^\circ) & = & \sin(1^\circ)\sin(91^\circ) \end{array}$$

and find that

$$\sin(45^\circ)\sin(45^\circ) + \sin(0^\circ)\sin(90^\circ) = \sin(91^\circ)\sum_{k=1}^{45}\sin((2k-1)^\circ) - \sum_{k=1}^{44}\sin(2k^\circ),$$

from which we can see that the expression in the problem statement is equal to $\sin(91^\circ) = \sin(89^\circ)$. Hence, the answer is $\boxed{89}$.

Problem 14 Let a, b, c, and d be the roots of the quartic polynomial $x^4 + x^3 + x^2 + 2$. Determine $|a|^2 + |b|^2 + |c|^2 + |d|^2$.

Answer: 6

Solution: Suppose there is a complex root with norm 1. In that case, the terms of the polynomial describe the sides of a quadrilateral with three of length 1 and one of length 2, with somewhat regular angles. Where have we seen such a quadrilateral before? It suggests half a regular hexagon! And, indeed, we can verify that $\cos(60) \pm \sin(60)i$ are roots. Factoring out the quadratic $x^2 - x + 1$, we get a quadratic whose roots can be found with the quadratic formula. Those roots turn out to be $-1 \pm i$. Hence the answer is $1 + 1 + 2 + 2 = \boxed{6}$.

Problem 15 The vertices of a tetrahedron are located at

$$(0, 0, 0),\ (10, 20, 30),\ (20, 20, 30), \text{ and } (30, 10, 10).$$

How many lattice points are there either inside the tetrahedron or on its boundary?

Answer: 286

Solution: Let $M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 3 & 1 \end{pmatrix}$. Notice that M has determinant 1 and its inverse is $\begin{pmatrix} -1 & 7 & -4 \\ 1 & -8 & 5 \\ 0 & 3 & -2 \end{pmatrix}$. This matrix, having determinant 1, maps the given tetrahedron to a tetrahedron that has the same number of lattice points in or on its boundary. The vertices map to $(0, 0, 0)$, $(10, 0, 0)$, $(0, 10, 0)$, and $(0, 0, 10)$, and the tetrahedron with these points as vertices has

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{12}{2} = \binom{13}{3}$$

lattice points in or on its boundary. Thus, the answer is $\binom{13}{3} = \boxed{286}$.

Problem 16 Let $S = \sum_{k=0}^{200} (-1)^k \binom{600}{3k}$. What is the prime factorization of S?

Answer: $2 \cdot 3^{299}$

Solution: Let $w = e^{2\pi i/3}$. Then

$$\sum_{k=0}^{200} (-1)^k \binom{600}{3k} x^{3k} = \frac{(1-x)^{600} + (1-wx)^{600} + (1-w^{-1}x)^{600}}{3}.$$

Substituting 1 for x, we see that $S = (1-w)^{600} + (1-w^{-1})^{600}$. Note that the norm of $1-w$ is $\sqrt{3}$ and its modulus is $-30°$. Thus, $S = 2 \cdot 3^{300}/3 = \boxed{2 \cdot 3^{299}}$.

Problem 17 Define the sequence y_n recursively as follows: $y_0 = 0$, $y_1 = 451$, and $y_{n+1} = 1560y_n - 901^2 y_{n-1}$, for $n > 0$. Determine the smallest positive integer N such that $y_N < 0$ and $y_{N+1} > 0$.

Answer: 11

Solution: Let $z_n = 901^{-n} y_n$. Then $z_{n+1} = 2(780/901)z_n - z_{n-1}$. In the right triangle with side lengths 780, 451, and 901, let A be the measure of the angle opposite the side of length 451, so that $\cos A = 780/901$. Note that $z_n = \sin(nA)$ since $z_0 = 0 = \sin(0)$ and $z_1 = 451/901 = \sin A$, and

$$\sin((n+1)A) + \sin((n-1)A) = 2\sin(nA)\cos(A),$$

so $\sin((n+1)A) = 2(780/901)\sin(nA) - \sin((n-1)A)$.

Thus, N is the largest positive integer k such that $kA < 2\pi$.

Now, $\sin A = 451/901 > \sin(\pi/6)$. Using concavity of sine on $[0, \pi/2]$, we compute that

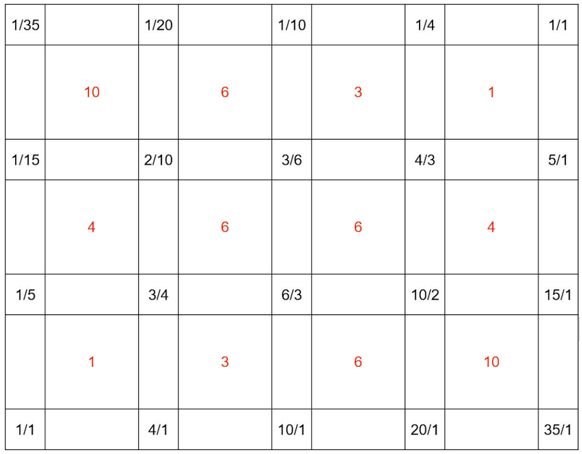
$$\sin\left(\frac{2\pi}{11}\right) = \sin\left(\frac{21}{22} \cdot \frac{\pi}{6} + \frac{1}{22} \cdot \frac{\pi}{2}\right) > \frac{21}{22}\sin\left(\frac{\pi}{6}\right) + \frac{1}{22}\sin\left(\frac{\pi}{2}\right) = \frac{23}{44} > \sin A.$$

Therefore $\frac{\pi}{6} < A < \frac{2\pi}{11}$ and $N = \boxed{11}$.

Problem 18 A rectangular gridwork of roads has 4 horizontal avenues crossed by 5 vertical streets. A vertical or horizontal road that connects adjacent intersections will be called a segment. Let P be the set of paths in the gridwork that begin in the upper left and end in the lower right and that do not go up or to the left. Let S_k be the number of ordered pairs (p, q), where $p, q \in P$, such that p and q share exactly k segments. It happens that $S_1 = 284$, $S_3 = 224$, and $S_4 = 100$. What is S_2?

Answer: 258

Solution: There are $\binom{7}{3} = 35$ paths total in P. If p and q share 6 or 7 segments, then they are the same path, therefore $S_6 = 0$ and $S_7 = 35$.



If p and q share 5 segments, then they can only differ in that there exists a single block where one path goes down its left side and along its bottom edge, whereas the other goes along its top edge then down its right edge. To compute S_5, for each block, we indicate in red in the above chart how many paths connect the top left of that block to its bottom right (the product of the number coming in and going out). (At each intersection, we give "A/B," where A is the number of paths from the upper left corner of the grid to that intersection and B is the number of paths from that intersection to the lower right corner of the grid.) We then add up twice these numbers (because we are working with ordered pairs) to get the total number of pairs: 120.

So far, we have

k	0	1	2	3	4	5	6	7
S_k	-	284	-	224	100	120	0	35

For each segment s, let P_s be the set of ordered pairs (p, q) of paths p, $q \in P$ that both use s.

Let n_k be the sum of the number of elements in a k-fold intersection of the P_s over all distinct (unordered) k-tuples of segments s. Using the principle of inclusion-exclusion, we have:

$$\begin{aligned}
S_7 &= n_7 \\
S_6 &= n_6 - 7n_7 \\
S_5 &= n_5 - 6n_6 + 21n_7 \\
S_4 &= n_4 - 5n_5 + 15n_6 - 35n_7 \\
S_3 &= n_3 - 4n_4 + 10n_5 - 20n_6 + 35n_7 \\
S_2 &= n_2 - 3n_3 + 6n_4 - 10n_5 + 15n_6 - 21n_7 \\
S_1 &= n_1 - 2n_2 + 3n_3 - 4n_4 + 5n_5 - 6n_6 + 7n_7
\end{aligned}$$

With existing data, we can solve for n_k, for $3 \le k \le 7$:

k	0	1	2	3	4	5	6	7
n_k	-	-	-	3049	1925	855	245	35

We also have

$$\begin{aligned} S_2 &= n_2 - 3207 \\ 284 &= n_1 - 2n_2 + 4497 \end{aligned}$$

To solve these equations, we compute n_1. In the chart below, within each segment, we indicate the number of paths that pass through that segment (in red). To find n_1, we add up the squares of these segment numbers.

1/35	20	1/20	10	1/10	4	1/4	1	1/1
15		10		6		3		1
1/15	10	2/10	12	3/6	9	4/3	4	5/1
5		8		9		8		5
1/5	4	3/4	9	6/3	12	10/2	10	15/1
1		3		6		10		15
1/1	1	4/1	4	10/1	10	20/1	20	35/1

Thus, $n_1 = 2717$, from which we find $n_2 = 3465$ and $S_2 = \boxed{258}$.

Problem 19 The sides of $\triangle ABC$ are rational and $AB = AC$. Inside $\triangle ABC$, two congruent circles of radius y are tangent to $\overline{BC}$ and, externally, to each other, with one tangent to $\overline{AB}$ and the other tangent to $\overline{AC}$. A circle of radius x is placed inside $\triangle ABC$ tangent to sides $\overline{AB}$ and $\overline{AC}$ and externally tangent to the circles of radius y. Given that x and y are integers, determine the smallest possible perimeter of the triangle whose vertices are the centers of the three circles.

Answer: 242

Solution: Let D be the center of the circle of radius x and let E and F be the centers of the circles of radius y. Let P and Q be the points of tangency of the circles of radius y with $\overline{BC}$. Let X be the point where the circle of radius y that is tangent to $\overline{AB}$ touches $\overline{AB}$ and let Y be the point where the circle of radius x touches $\overline{AB}$.

Since $\overline{BC}$ is rational and $PQ = 2y$, we know that $BP = BX$ is rational. Therefore, $\tan(m\angle ABC/2) = \frac{y}{BP}$ is rational. Hence, $\tan(m\angle ABC)$ is rational. Therefore, $\tan(m\angle BAC/2) = 1/\tan(m\angle ABC)$ is rational, and hence AY is rational. Since BX and AY are rational, so is XY. Now XY is the length of the leg of a right triangle with hypotenuse $x + y$ and other leg of length $|x - y|$. Therefore, XY must be an integer.

Since $\tan(m\angle ABC)$ is rational, the altitude of $\triangle ABC$, with respect to base $\overline{BC}$, is rational, and therefore, the sine, cosine, and tangent of $m\angle BAC/2$ are all rational. Hence AD is rational, and therefore the height of $\triangle DEF$, with respect to $\overline{EF}$ as base, is rational, and since this altitude is the leg of a right triangle with other leg length y and hypotenuse $x+y$, it is, in fact, an integer.

Therefore, $\overline{DE}$ is the hypotenuse of two Pythagorean triples.

If we parameterize the Pythagorean triples corresponding to the right triangle obtained by bisecting $\triangle DEF$ along the perpendicular bisector of $\overline{EF}$ by (m, n) with $m > n > 0$ and m and n relatively prime and of opposite parity, the situation we have is that $m^2 + n^2$ must also be the hypotenuse of a Pythagorean triple with one leg of length $|m^2 + n^2 - 2l|$, where l is either $m^2 - n^2$ or $2mn$. However, l cannot be $m^2 - n^2$ since that would imply $(m^2 - n^2)(2n^2)$ is a perfect square, which is impossible. Therefore l must be $2mn$ and we must have that $2mn$ be a perfect square.

We also need that $1/4 < x/y < 2$ for outside of this range, the three circles cannot be placed inside an isosceles triangle as described in the problem. This translates to $3 - \sqrt{2} < n/m < 1/2$. We seek to minimize the perimeter of $\triangle DEF$, which is $2(m + n)^2$. This occurs when $m = 9$ and $n = 2$, yielding a perimeter of $2(9 + 2)^2 = \boxed{242}$.

Problem 20 Define the sequence d_n recursively by $d_1 = 1$ and

$$d_{n+1} = \frac{48\sqrt{n} - 7}{48\sqrt{n+1} + 7} d_n,$$

for positive integers n. Evaluate

$$\sum_{k=1}^{\infty} d_k \,.$$

Express your answer as a fraction in simplest form.

Answer: $\dfrac{55}{14}$

Solution: Suppose x_n is a sequence that satisfies $x_{n+1} = \frac{48\sqrt{n}-7}{48\sqrt{n}+7}x_n$. Rearranging terms, we find that

$$48\sqrt{n}(x_n - x_{n+1}) - 7x_n - 7x_{n+1} = 0.$$

Also,

$$48\sqrt{n+1}(x_{n+1} - x_{n+2}) - 7x_{n+1} - 7x_{n+2} = 0.$$

Subtracting the first from the second and substituting c_n for $x_n - x_{n+1}$, we get

$$48\sqrt{n+1}c_{n+1} + 7c_n + 7c_{n+1} - 48\sqrt{n}c_n = 0.$$

Rearranging, we have

$$c_{n+1} = \frac{48\sqrt{n} - 7}{48\sqrt{n+1} + 7}c_n,$$

which is the same recursion relation satisfied by the sequence d_n.

If we can arrange things so that $c_1 = d_1 = 1$, then we would have $c_n = d_n$ for all $n \geq 1$. So, we insist that $c_1 = x_1 - x_2 = x_1 - \frac{41}{55}x_1 = \frac{14}{55}x_1 = 1$, or $x_1 = \frac{55}{14}$.

By doing this, we find that

$$\sum_{k=1}^{\infty} d_k = \sum_{k=1}^{\infty}(x_k - x_{k+1}) = x_1 - \lim_{k\to\infty} x_k.$$

We claim that $\lim_{k\to\infty} x_k = 0$. Note that $x_k > 0$ and the sequence x_k is strictly decreasing, so its limit exists. Let $z_n = x_{n^2}$. Then

$$z_{n+1} < \left(\frac{48(n+1) - 7}{48(n+1) + 7}\right)^{2n+1} z_n = \left(1 - \frac{14}{48n + 55}\right)^{2n+1} z_n.$$

Now,

$$\left(1 - \frac{14}{48n + 55}\right)^{2n+1} < e^{-(28n+14)/(48n+55)} < e^{-42/103} < 1,$$

where the second inequality follows from the fact that $-\frac{28n+14}{48n+55}$ is strictly decreasing for $n \geq 1$. Therefore, z_n is bounded above by a geometric sequence with a common ratio that is less than 1, and hence, both z_n and x_n tend to 0 as n tends to ∞.

We conclude that the answer is $x_1 = \boxed{\dfrac{55}{14}}$.